February 1965

USCEE Report 129

UNIVERSITY OF SOUTHERN CALIFORNIA

SCHOOL OF ENGINEERING

SIGNAL DESIGN FOR SPACE COMMUNICATION CHANNELS
PART I

Charles L. Weber

COMMUNICATION SYSTEMS RESEARCH LABORATORY

Department of Engineering
University of Southern California
Los Angeles, California

ELECTRONICS SCIENCES LABORATORY

USC Engineering

February 1965

SIGNAL DESIGN FOR SPACE COMMUNICATION CHANNELS

PART I

Charles L. Weber

COMMUNICATION SYSTEMS RESEARCH LABORATORY

Department of Engineering
University of Southern California
Los Angeles, California

FOREWORD

The research described in this report, "Signal Design for Space Communication Channels," by Charles L. Weber, was conducted under the joint sponsorship of the Space Science Center at the University of California, Los Angeles, under NASA Contract Number NsG 237/62, and the University of Southern California by the Joint Services Electronics Program (U.S. Army, U.S. Navy, and U.S. Air Force) under Grant Number AF-AFOSR-496-65, and is part of a continuing program.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

I. INTRODUCTION

The fundamental purpose of communication and telemetry systems is the transmission of reliable information or data through an unreliable channel. In space communications a major constraint is the limitation of transmitter power available on the spacecraft. Or equivalently, for a given transmitter size, and hence power, we are interested in maximizing the possible range that the spacecraft can attain while still maintaining telemetry transmission to the earth within a specified error rate. Also, the waveforms of the transmitted signals are limited by the complexity of the coding and transmitting equipment, but more significantly by their allowed bandwidth. The bandwidth normally must be small enough so that transmission over adjacent channels is not affected by the operation in the given channel.

As early as 1949, Shannon [1. 1] has shown that it should be possible to exchange bandwidth for power, so that by increasing bandwidth one should, at least in principle, be able to make up for the lack of transmitter power. This can be made slightly more precise by introducing channel capacity, as derived by Shannon [1. 1] for a given bandwidth B, namely

$$C = B \log_2 \left[1 + \frac{P}{N} \right] \tag{1.1}$$

where P is the signal power and N is the noise power. When the communication is that of transmission and reception of electromagnetic signals, whether microwave or optical, between a satellite or a deep space vehicle and the earth, the channel disturbances can be represented by additive white Gaussian noise. This is because the principle source of disturbances in space communications is Galactic noise, which has a very wide bandwidth and near constant spectral density, thus representable as white noise. The advent of the maser has reduced internal thermal noise in the microwave region to where it is negligible in comparison to Galactic noise. Similarly in optical systems, using lasers for instance, the limiting noise due to the "zero-point field" is again white Gaussian. The noise spectral density in either case is given by

$$\Phi(\nu) = \frac{h\nu}{2} + \frac{h\nu}{e^{h\nu/kT_s} - 1} \tag{1.2}$$

where T_s is the source temperature, ν the carrier frequency, h Plank's constant, and k Boltzman's constant. Thus, with the advent of space communications, we have "real-life" channels which can be statistically described by additive Gaussian noise.

Hence, if the noise spectral density is Φ, then the noise power is

$$N = (\Phi)(2B) \tag{1.3}$$

and

$$C_B = B \log_2 \left[1 + \frac{P}{2\Phi B}\right] \tag{1.4}$$

which is a maximum when B increases without bound. The maximum value is

$$C_\infty = \frac{P}{2\Phi} \log_2 e. \tag{1.5}$$

By Shannon's coding theorems, for a given information rate H, we can transmit at rate H with zero error if

$$H < C. \tag{1.6}$$

Hence, if bandwidth is no limitation, we should be able to attain transmission with no error if

$$H < \frac{P}{2\Phi} \log_2 e \tag{1.7}$$

or the power required need not be greater than

$$\frac{2H\Phi}{\log_2 e} \tag{1.8}$$

However, there are two major drawbacks in this theory. For one thing, the coding theorems do not tell us how to attain zero error transmission. Secondly, even if we were fortunate enough to find the optimum code, we know that it would require transmission signals of infinite time duration to attain zero error. In a practical system, however, the time duration T of the signal waveforms is fixed and finite. Moreover, regardless of which concept of bandwidth we use, infinite bandwidth is never allowed in a practical system.

The approach we take here is to design optimal signals when they are constrained to a specified average power P and a finite time duration T. We shall state precisely what we mean by finite bandwidth and how this affects optimal signal design. We shall adopt the design criterion of probability of error because of its inherent physical significance based on the law of large numbers. The optimization can then be viewed either as minimizing the average power for a given probability of error or as minimizing the probability of error for a given allowable signal power. The results will be the same. We employ the latter optimization here. This places the optimal signal design problem in the framework of detection theory, differing from most previous approaches in that they placed the problem in the context of information theory or error-correcting codes. Although the information-theoretic notions are not necessary in what follows, we shall often find them useful, if only to indicate what the connections are.

REFERENCES:

1.1 C. E. Shannon, The Mathematical Theory of Communication, University of Illinois Press, 1949.

APPENDIX A

SUMMARY OF CONDITIONAL GAUSSIAN PROBABILITY DENSITY FUNCTIONS

Let x be an n-dimensional Gaussian variate with mean u.

Let y be an m-dimensional Gaussian variate with mean v.

Let

$$Z = \begin{pmatrix} x \\ y \end{pmatrix},$$

which has Gaussian probability density function

$$G\left(Z; \begin{pmatrix} u \\ v \end{pmatrix}; V\right)$$

where

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

where

V_{11} is the covariance matrix of x,

V_{22} is the covariance matrix of y,

and $V_{12} = V_{21}^{*}$ is the correlation matrix between x and y.

Then the conditional density of x given y, is

$$f(x/y) = G(x;\beta;C)$$

where

$$\beta = u + V_{12}V_{22}^{-1}(y - v)$$

and

$$C = V_{11} - V_{12}V_{22}^{-1}V_{21}$$

If y is a singular density, it is sufficient to consider the largest number of components of y whose density is non-singular, neglecting the other components.

APPENDIX B

SUMMARY OF TETRA-CHORIC SERIES

(See Reference 4.7 for a more complete development of the following Theorems.)

Theorem B.1. The Bivariate Gaussian Distribution

Let F be a bivariate Gaussian distribution with zero means, unit variances, and correlation coefficient ρ.

Define

$$d = \int_h^\infty \int_k^\infty dF$$

Then

$$d = \sum_{r=o}^{\infty} \frac{\rho^r}{r!} \tau_r(h)\, \tau_r(k)$$

where

$$\tau_r(x) = \left(-\frac{d}{dx}\right)^{r-1} G(x),$$

$$G(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2},$$

and the convergence is uniform for $|\rho| \leq 1$.

This is called the tetra-choric series for the bivariate Gaussian distribution.

Theorem B.2. For the M-dimensional case:

$$d = \int_{h_1}^\infty \cdots \int_{h_M}^\infty G(x; 0; \alpha)\, d|x|$$

where

$$\alpha = \begin{pmatrix} 1 & & \lambda_{ij} \\ & \ddots & \\ \lambda_{ji} & & 1 \end{pmatrix}$$

Then

$$d = \sum_{r=o}^{\infty} \left\{ \underbrace{\sum_{n_{12}} \cdots \sum_{n_{ij}} \cdots \sum_{n_{M-1,M}}}_{\text{such that } \sum\sum_{i>j} n_{ij} = r} \left[\prod_{i>j}\prod \frac{(\lambda_{ij})^{n_{ij}}}{(n_{ij})\,!} \right] \left[\prod_{k=1}^{M} \left(-\frac{d}{dx}\right)^{q_k - 1} G(h_k) \right] \right\}$$

where,

$$\sum_{\substack{i=1 \\ i \neq j}}^{M} n_{ij} = \sum_{\substack{i=1 \\ i \neq j}}^{M} n_{ji} = q_j$$

Note:

1) $\{n_{ij}\}$ and $\{q_j\}$ depend on the value of r.

2) The r = 0 term is

$$\prod_{i=1}^{M} \int_{h_i}^{\infty} G(x)\, dx$$

3) If we wish to expand

$$d' = \int_{-\infty}^{h_1} \cdots \int_{-\infty}^{h_M} G(x; 0; \alpha)\, d|x|$$

instead of d, the summation $\sum_{r=1}^{\infty}$ remains unchanged, but the r=0 term becomes

$$\prod_{i=1}^{M} \int_{-\infty}^{h_i} G(x)\, dx = \prod_{i=1}^{M} \phi(h_i)$$

II. PROBLEM STATEMENT FOR COHERENT CHANNELS

2.1 Description in the Time Domain

We now describe the kind of communication link with which we will be concerned. A basic block diagram of the system is shown in Figure 2.1.

The information or data usually comes from several analog sources which are sampled, digitized, and arranged in the form of sequences of binary digits, although in general, the digitized symbols could be elements from a K-ary alphabet. The encoder maps sequences of digits of length n one-to-one onto a set of M time varying waveforms each of T seconds duration. This is known as block encoding. This resulting sequence of time varying signals is then used to modulate a high frequency carrier (AM, FM, PM, etc., or combinations of these). Since each member of each sequence can be any element of the alphabet, we necessarily have $M = (K)^n$, or $M = 2^n$ in the binary case. Thus, the primary source may be taken as discrete having a nominal rate of H bits per second. If the source is binary, as is usually the case, then the maximum possible rate is of course the number of bits per second, and this is what is normally taken as H, since the system must certainly be prepared to handle this rate. Because it is normally impossible to specify quantitatively the precise probabilistic structure of the source, the maximum value of H is assumed, namely

$$H = \frac{1}{T} \log_2 M$$

Thus in one T second interval, the total number of possible signals is

$$M = 2^{HT}$$

It is clear that M and T are more basic to us than H. For our purposes, we could equivalently say that the M transmittable waveforms are equi-likely and that each successive waveform is independent of all previous ones, neglecting the form of the data source and the encoder.

The transmitted signal for a particular T second interval is assumed to be of the form

$$V\, S_j(t)$$

when V is an amplitude scale factor, $S_j(t)$ is one of M equi-likely signals of the form

$$S_j(t) = A_j(t) \cos\left(\omega_c t + \phi_j(t)\right)$$
$$0 \le t \le T, \qquad j = 1, \ldots, M; \tag{2.1}$$

$A_j(t)$ and $\phi_j(t)$ are narrow band signals with respect to the radian carrier frequency ω_c; and the $A_j(t)$ are normalized, i.e.,

$$\frac{1}{T}\int_0^T A_j^2(t)\, dt = 1 \qquad j = 1, \ldots, M \tag{2.2}$$

so that $\frac{V^2}{2}$ is the average power level of the transmitted waveform. Note that a completely equivalent way to write the transmitted signals would be to introduce complex waveforms, as in Reference 2.1.

The received signal for a particular T period will be denoted by

$$y(t) = V\, S_j(t) + N(t) \qquad 0 \le t \le T, \tag{2.3}$$

where N(t) is white Gaussian noise with two-sided spectral density Φ_c.

The receiver is assumed to have the following characteristics:

(1) It knows the form (except for the amplitude V) of all the transmitted signals, i.e., knows the form of each $S_j(t)$, $j = 1, \ldots, M$.

(2) It is synchronized in time (i.e., knows the time interval [O, T] during which the signal will arrive), and is phase coherent (i.e., knows the carrier phase angle). The recent development of synchronous codes and phase locked loops makes these two assumptions physically attainable.

(3) Its sole purpose is to decide at time T which one of the M signals $S_j(t)$ has been transmitted, based only on the waveform received during the interval $[O, T]$.

For a fixed M-ary signaling alphabet $\{S_j(t)\}$, and using maximum probability of detection as the optimization criterion, it is readily shown that the optimum receiver is the one which forms the following scalar products (see Figure 2.2):

$$E_i = \left[y(t) \cdot S_i(t) \right] = \int_O^T y(t)\, S_i(t)\, dt \tag{2.4}$$

$$i = 1, \ldots, M$$

and decides that the j^{th} signal has been transmitted if

$$E_j = \max \left(E_1, E_2, \ldots, E_M \right)^* \tag{2.5}$$

The notation used in Equation (2.4) for the scalar product (or correlation of time functions) will be used throughout.

For every set of M signals of the form specified by Equation (2.1), there exists an optimum receiver and a corresponding probability of detection. In this large class of sets of M signals, all restricted to an energy of $\frac{T\,V^2}{2}$ in T seconds, we wish to find that subclass which has the largest probability of detection. We also want to determine whether this optimum class of sets is independent of the average power level $\frac{V^2}{2}$. The following section will indicate that determining this subclass can be reduced to finding certain classes of D-dimensional vectors on the D-dimensional unit sphere satisfying certain optimization requirements.

*Note that the decision rule will not be altered if all the E_i's are multiplied by a common positive scale factor.

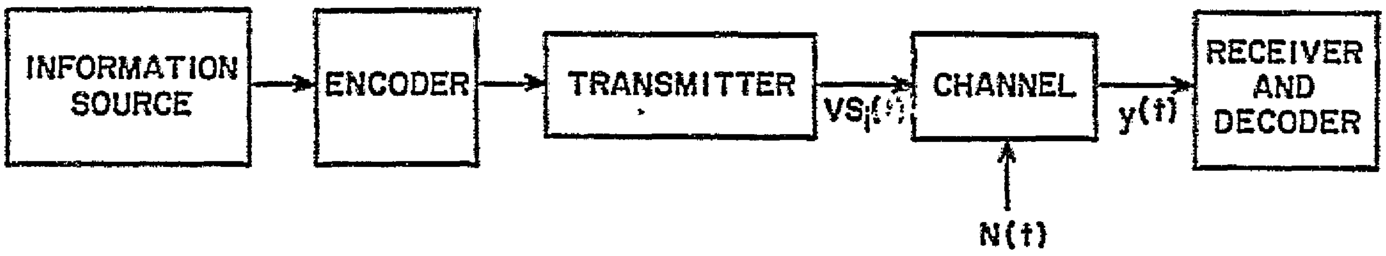


FIGURE 2.1

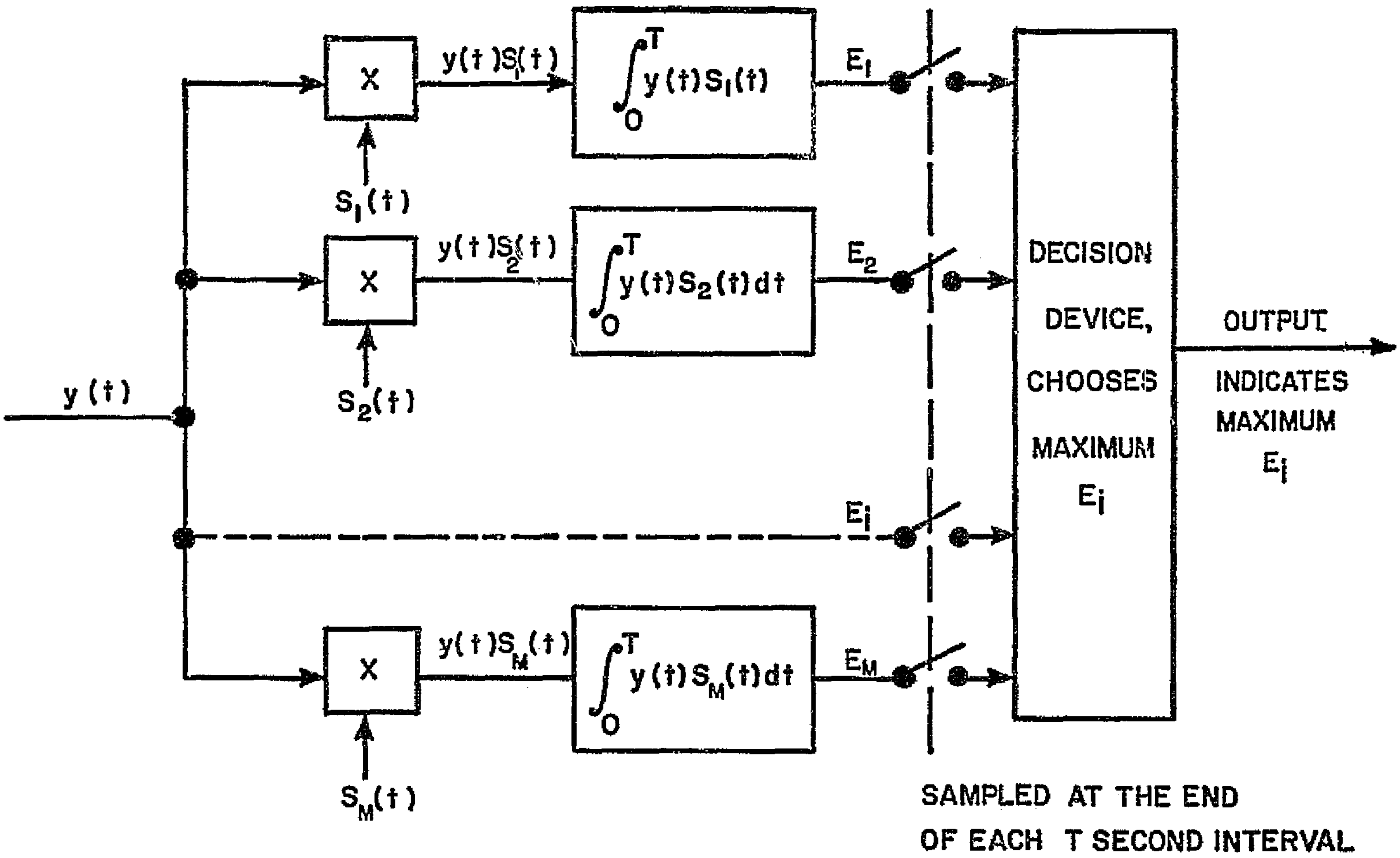


FIGURE 2.2

2.2 Reduction to Finite Dimensional Euclidean Space

Using (2.2), the average received signal power is given by

$$P_{av} = \frac{1}{T}\int_0^T \left[V S_j(t)\right]^2 dt = \frac{V^2}{2T}\int_0^T A_j^2(t)\, dt$$
$$+ \frac{V^2}{2T}\int_0^T A_j^2(t) \cos\left(2\omega_c t + 2\phi_j(t)\right) dt = \frac{V^2}{2} \qquad (2.6)$$

assuming that the carrier frequency is high enough to neglect the second integral. Integrals of this form will be omitted throughout, and is justified on account of the narrow-banded assumption of $A_j(t)$ and $\phi_j(t)$. The signal energy received during the interval $[O, T]$ is

$$\text{Energy} = T P_{av} = \frac{V^2 T}{2} \qquad (2.7)$$

From (2.3) and (2.4), the i^{th} correlator output, E_i, assuming the j^{th} signal has been transmitted, is given by

$$E_i = \int_0^T \left[V S_j(t) + N(t)\right] S_i(t)\, dt$$
$$= V\int_0^T A_j(t)\, A_i(t) \cos\left(\omega_c t + \phi_i(t)\right) \cos\left(\omega_c t + \phi_j(t)\right) dt$$
$$+ \int_0^T N(t)\, A_i(t) \cos\left(\omega_c t + \phi_i(t)\right) dt$$
$$i = 1, \ldots, M. \qquad (2.8)$$

Expanding and applying the narrow-banded assumption, E_i can be written as

$$E_i = \int_0^T \left[\frac{V}{2} A_j(t) \cos\phi_j(t)\right]\left[A_i(t) \cos\phi_i(t)\right] dt$$
$$+ \int_0^T \left[\frac{V}{2} A_j(t) \sin\phi_j(t)\right]\left[A_i(t) \sin\phi_i(t)\right] dt$$
$$+ \int_0^T \left[N(t) \cos\omega_c t\right]\left[A_i(t) \cos\phi_i(t)\right] dt$$
$$- \int_0^T \left[N(t) \sin\omega_c t\right]\left[A_i(t) \sin\phi_i(t)\right] dt$$
$$i = 1, \ldots, M \qquad (2.9)$$

By letting $\overline{S}_j(t)$ be the two dimensional time vector

$$\overline{S}_j(t) = \begin{pmatrix} S_{j1}(t) \\ S_{j2}(t) \end{pmatrix} = \begin{pmatrix} A_j(t)\cos\phi_j(t) \\ A_j(t)\sin\phi_j(t) \end{pmatrix}$$

$$0 \leq t \leq T, \quad j = 1, \ldots, M \tag{2.10}$$

and specifying $\overline{N}(t)$ and $\overline{y}(t)$ by

$$\overline{N}(t) = \begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} = \begin{pmatrix} N(t)\cos\omega_c t \\ -N(t)\sin\omega_c t \end{pmatrix} \tag{2.11}$$

and

$$\overline{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \frac{V}{2}\overline{S}_j(t) + \overline{N}(t) \tag{2.12}$$

the E_i and corresponding probability of detection can be computed by considering the following equivalent system:

Let

$\overline{S}_j(t)$ be transmitted signal vector,

$\overline{N}(t)$ be the additive noise vector,

and $\overline{y}(t)$ the received signal vector.

The optimum receiver in this case performs the following operations:

$$E_i = \left[\overline{y}(t) \cdot \overline{S}_i(t)\right] = \left[y_1(t) \cdot S_{i1}(t)\right] + \left[y_2(t) \cdot S_{i2}(t)\right] \quad i = 1, \ldots, M \tag{2.13}$$

which agrees with (2.9). Thus an observer at the receiver output would not be able to distinguish between the two systems, since the set of E_i's is identical for each.

Furthermore it is noticed that the carrier frequency has been eliminated from the signal set in the second model. Because there exists a one-to-one correspondence between the signal sets of the two models, an optimum set for one model corresponds to an optimum set for the other. Since the signal set in the second model does not involve the carrier frequency, it has been eliminated from the optimization, provided it is sufficiently large to make the narrow-banded assumptions.

A further reduction is possible to finite-dimensional vectors, since the signal set is finite. That is, we can write

$$\bar{S}_j(t) = \sum_{k=1}^{D} s_k^i \; \bar{\psi}\, k(t), \qquad j = 1, \ldots, M \tag{2.14}$$

where

$$\bar{\psi}_k(t) = \begin{pmatrix} \psi_{k1}(t) \\ \psi_{k2}(t) \end{pmatrix} \qquad k = 1, \ldots, D \tag{2.15}$$

is an orthogonal set of basis functions such that

$$\left[\bar{\psi}_k(t) \cdot \bar{\psi}_\ell(t) \right] = \left[\psi_{k1}(t) \cdot \psi_{\ell 1}(t) \right] + \left[\psi_{k2}(t) \cdot \psi_{\ell 2}(t) \right] \tag{2.16}$$

$$= T \; \delta_{k\ell}$$

and where

$$s_k^i = \frac{1}{T} \left[\bar{S}_i(t) \cdot \bar{\psi}_k(t) \right] \tag{2.17}$$

The dimensionality D of the signal set is at most M, and is a measure of the bandwidth of the signal set. This will be discussed in detail in the next section.

From (2.14), each signal can now be characterized by the D-dimensional vector

$$\bar{S}_i = \begin{pmatrix} s_{i1} \\ \vdots \\ s_{ij} \\ \vdots \\ s_{iD} \end{pmatrix} \qquad i = 1, \ldots, M \tag{2.18}$$

The noise vector $\bar{N}(t)$ can also be expanded using the orthogonal basis functions, i.e., express $\bar{N}(t)$ as

$$\bar{N}(t) = \sum_{j=1}^{D} \xi_j \, \bar{\psi}_j(t) + \Delta(t) \tag{2.19}$$

where the ξ_j are Gaussian random variables with

$$E(\xi_j) = \frac{E\left\{\left[\bar{N}(t) \cdot \bar{\psi}_j(t)\right]\right\}}{T} = 0 \tag{2.20}$$

and

$$E\left(\xi_i \xi_j\right) = \frac{1}{T^2} \, E\left(\left[\bar{N}(\alpha) \cdot \bar{\psi}_i(\alpha)\right] \left[\bar{N}(\beta) \cdot \bar{\psi}_j(\beta)\right]\right)$$

$$= \frac{\Phi_c}{2T} \, \delta_{ij} \tag{2.21}$$

and where $\Delta(t)$ is that part of the noise which is orthogonal to the basis functions $\bar{\psi}_j(t)$, $j = 1, \ldots, D$, i.e.,

$$E\left(\left[\Delta(t) \cdot \bar{\psi}_j(t)\right]^2\right) = 0 \qquad j = 1, \ldots, D. \tag{2.22}$$

Since $\Delta(t)$ is orthogonal to each $\bar{\psi}_j(t)$, $\Delta(t)$ does not affect the value of any of the correlator outputs E_i, $i = 1, \ldots, M$. Therefore, for our optimization purposes, $\Delta(t)$ can be eliminated from consideration, and if we define

$$\bar{N}_o(t) = \sum_{i=1}^{D} \xi_i \, \bar{\psi}_i(t) \tag{2.23}$$

we can allow the received signal $\bar{y}(t)$ to be

$$\bar{y}(t) = \frac{V}{2} \bar{S}_j(t) + \bar{N}_o(t). \tag{2.24}$$

$\bar{N}_o(t)$ can be characterized by the D-dimensional noise vector

$$\bar{Z} = \begin{pmatrix} \xi_1 \\ \cdot \\ \cdot \\ \cdot \\ \xi_D \end{pmatrix} \tag{2.25}$$

Substituting each of these expansions into (2.13), we can express E_i as

$$\begin{aligned} E_i &= \left[\left(\sum_{h=1}^{D} \left(\frac{V}{2} s_k^j + \xi_k \right) \bar{\psi}_k(t) \right) \cdot \left(\sum_{\ell=1}^{D} s^i_\ell \, \bar{\psi}_\ell(t) \right) \right] \\ &= T \left(\frac{V}{2} \bar{S}_j + \bar{Z} \right)^* \bar{S}_i \qquad\qquad i = 1, \ldots, M \end{aligned} \tag{2.26}$$

where the asterisk means "the transpose of". Also

$$E(\bar{Z}) = 0,$$

and

$$E(\bar{Z}\,\bar{Z}^*) = \frac{\Phi_c}{2T}(I) \tag{2.27}$$

where I is the DxD identity matrix.

We shall designate the signal vector inner products by

$$\bar{S}_i^{*} \bar{S}_j = \sum_{k=1}^{D} s_k^i s_k^j = \begin{cases} 1 & \text{if } i=j \\ \lambda_{ij} & \text{if } i \neq j \end{cases}$$

The $\bar{S}_j$ are vectors on the D-dimensional unit sphere.

From (2.26) it is noticed that the E_i can be formed by assuming the transmitted signal is the D-dimensional vector $\frac{V}{2} \bar{S}_j$, the channel adds the noise vector $\bar{Z}$, and the received signal is the vector

$$\bar{V} = \frac{V}{2} \bar{S}_j + \bar{Z}$$

The optimum receiver forms the scalar products indicated by (2.26) and again decides $\bar{S}_j$ was transmitted if

$$E_j = \max_i E_i \tag{2.5}$$

The decision rule and therefore the probability of detection will not be affected if the E_i are all multiplied by a scale factor. This allows us to multiply the received vector by the factor $\sqrt{\frac{2T}{\Phi_c}}$, and to neglect the T in (2.26). Then we can define a normalized received vector

$$\bar{V}_o = \frac{V\sqrt{T}}{\sqrt{2\,\Phi_c}} \bar{S}_j + \bar{Z}_o \tag{2.28}$$

where

$$\bar{Z}_o = \sqrt{\frac{2T}{\Phi_c}}\, \bar{Z}$$

Here $\bar{Z}_o$ is a D-dimensional Gaussian noise vector with zero mean and covariance matrix equal to the DxD identity matrix.

Let

$$\lambda = \frac{V\sqrt{T}}{\sqrt{2\,\Phi_c}} \tag{2.29}$$

Then

$$\bar{\bar{V}}_o = \lambda \bar{S}_j + \bar{\bar{Z}}_o \tag{2.30}$$

λ^2 is the signal energy to noise spectral density ratio, which we will henceforth call the signal-to-noise ratio (SNR).

This last formulation of the E_i's cannot be distinguished (except for a scale factor) from the basic model formulation by an observer stationed at the receiver output. Using (2.30) and (2.5), the probability of detection P_D is given by

$$P_D = \sum_{j=1}^{M} Pr(\bar{S}_j)\, Pr\left[(\bar{V}_o^{\,*}\, \bar{S}_j) = \max_i (\bar{V}_o^{\,*} \bar{S}_i) \Big/ \bar{S}_j \text{ was transmitted}\right]$$

or equivalently

$$P_D = \frac{1}{M} \sum_{j=1}^{M} Pr\left[E_j = \max_i E_i \Big/ \bar{S}_j \text{ was transmitted}\right] \tag{2.31}$$

The problem of determining the set of $\{\bar{S}_j\}$ which maximizes P_D for various M and D and of determining the dependence or independence of these optimal sets on SNR is known as the sphere-packing problem of Communication Theory. The solution of this problem (also known as the signal selection problem) is our primary goal.

In this section we have shown that the signal design problem can be reduced to that of finding M unit vectors on a D-dimensional unit sphere which maximize a given functional, namely probability of detection.

We conclude the section by indicating that once the optimum set of $\{\bar{S}_j\}$ has been determined, the corresponding set of $\{A_i(t)\}$ and $\{\phi_i(t)\}$ can easily be determined. Using the optimum set of $\{\bar{S}_j\}$

$$\begin{pmatrix} A_j(t) \cos \phi_j(t) \\ \\ A_j(t) \sin \phi_j(t) \end{pmatrix} = \sum_{i=1}^{D} s_i^j\, \bar{\psi}_i(t) = \begin{pmatrix} a_j(t) \\ \\ b_j(t) \end{pmatrix}$$

where

$$a_j(t) = \sum_{i=1}^{D} s_i^j \, \psi_{i1}(t)$$

and

$$b_j(t) = \sum_{i=1}^{D} s_i^j \, \psi_{i2}(t)$$

Then

$$A_j(t) = \sqrt{a_j^2(t) + b_j^2(t)}$$

and

$$\phi_j(t) = \text{principal value of } \tan^{-1} \frac{b_j(t)}{a_j(t)}$$

2.3 Bandwidth Considerations

In Chapter IV, we will show that the probability of detection can be written as a function of only SNR and the set of inner products $\{\lambda_{ij}\}$ of the signal vectors. That is, the only characteristic of the basis functions used in determining the probability of detection is that they are orthogonal. The actual waveshapes do not enter in. This means that if a second communication system is considered, which has a different transmittable signal set $\{S_i'(t), \ i = 1, \ldots, M\}$ which can be expressed in terms of a different orthogonal basis, say $\{\Gamma_i(t), \ i = 1, \ldots, D\}$, but with the same linear combinations as those in the original system, i.e.,

$$S_i'(t) = \sum_{j=1}^{D} s_j^i \, \Gamma_j(t), \quad i = 1, \ldots, M \tag{2.32}$$

where

$$\left[\Gamma_i(t) \cdot \Gamma_j(t) \right] = T\delta_{ij}$$

and with M, D, V, T, Φ_c, and the set of $\{s_j^i\}$ the same in both systems, then the probability of detection for both systems is equal. Or, more generally speaking, the probability of detection is a function of λ, M, D, and the set of signal vector inner products $\{\lambda_{ij}\}$, and depends not at all on which orthogonal basis is used to form the signal waveforms.

Therefore, the D basis functions can be designed or chosen according to some other criterion, and can be done so independently of the signal selection problem. The criterion normally used is to choose the basis functions to conform to some specified bandwidth restrictions. The only parameter which affects both the selection of optimal signal vectors and the selection of the orthogonal basis functions is the total number of basis functions, D.

A realistic way to arrive at a value for D is the following:

For a given time interval [O,T] outside of which the waveforms must be identically zero, how many orthogonal waveforms can I design which have a given percent energy within a specified bandwidth [-B,B], realizing, as will be shown later, that the maximum value D need be is M-1?

If D has already been determined because of its effect on probability of error, which will also be discussed in detail, then the criterion for determining basis functions could be rephrased in either of the following ways:

1. For given T and D, what is the minimum bandwidth, B, required for D orthogonal functions which vanish in time outside [O, T] and have a given percent energy within the bandwidth [-B, B]?

2. For given T and D, what is the largest percent of total energy that D orthogonal functions can have within the given bandwidth [-B, B], when the functions vanish in time outside [O, T]?

The problem of determining these optimal waveforms when the criterion is any of the above is discussed by Slepian and Pollak[2.2], and Landen and Pollak.[2.3]

The main point is that, except for the dimensionality D, the selection of signal vectors to minimize probability of error is disjoint from the problem of choosing waveforms that conform to certain bandwidth restrictions.

For a given T and given percent energy restriction, the bandwidth required for D orthogonal waveforms that satisfy the energy requirement will increase as D increases. For this reason D is said to be a measure of the required bandwidth of the signal waveforms.

REFERENCES

2.1 Balakrishnan, A.V., Advances in Communication Systems, McGraw-Hill, 1965.

2.2 D. Slepian and H.O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty-I", BSTJ, January, 1961, pp. 43-64.

2.3 H.J. Landau and H.O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty-II", BSTJ, January, 1961, pp. 65-84.

2.4 Balakrishnan, A.V., "Estimation and Detection Theory for Multiple Stochastic Processes", Journal of Math. Analysis and Applications, Vol. 1, pp. 386-410, 1960.

III. SIGNAL DESIGN WHEN THE DIMENSIONALITY OF THE SIGNAL SET IS RESTRICTED TO TWO

We now study the optimal signal design problem for the case when the dimensionality of the signal set is restricted to two. The signal set consisting of equal spacing of the M vectors around the unit circle is proven to be the optimum and this optimal choice is shown to be independent of signal-to-noise ratio. The question is then raised as to whether it is possible to have two suboptimal signal sets, say $\{\bar{S}_i,\ i = 1, \ldots, M\}$ and $\{\bar{S}_i',\ i = 1, \ldots, M\}$, such that the probability of detection for $\{\bar{S}_i\}$, namely $P_D\left(\lambda; \{\bar{S}_i\}\right)$, is larger than that for $\{\bar{S}_i'\}$ for some SNR while the reverse is true for other SNR. A subclass of the two dimensional signal sets is shown not to possess this property. However, by example, it is shown that there are signal sets such that the preference of one to the other does indeed depend on the signal-to-noise ratio.

When the signal waveforms are restricted to two dimensions, they can be expressed in the time domain as

$$\bar{S}_j(t) = \begin{pmatrix} A_j(t) \cos \phi_j(t) \\ A_j(t) \sin \phi_j(t) \end{pmatrix} = s_1^j\ \bar{\psi}_1(t) + s_2^j\ \bar{\psi}_2(t)$$

$$j = 1, \ldots, M \tag{3.1}$$

One choice of basis functions are those which are non-time varying, namely

$$\bar{\psi}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{\psi}_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ 0 \le t \le T \tag{3.2}$$

Any choice of basis functions is possible of course, but in the two dimensional case these are the most logical. With this choice $\bar{S}_j(t)$ is then non-time varying. Hence, $\bar{S}_j(t)^* \bar{S}_j(t)$ is non-time varying, which indicates $A_j(t)$ and therefore $\phi_j(t)$ are constants for each j. Since

$$\frac{1}{T}\int_0^T A_j^2(t)\ dt = 1, \qquad j = 1, \ldots, M$$

and since $A_j(t) = A_j$, we get $A_j = 1$, $j = 1, \ldots, M$. Therefore,

$$\overline{S}_j(t) = \begin{pmatrix} \cos\phi_j \\ \sin\phi_j \end{pmatrix} = \begin{pmatrix} s_1^j \\ s_2^j \end{pmatrix} \qquad j = 1, \ldots, M. \tag{3.3}$$

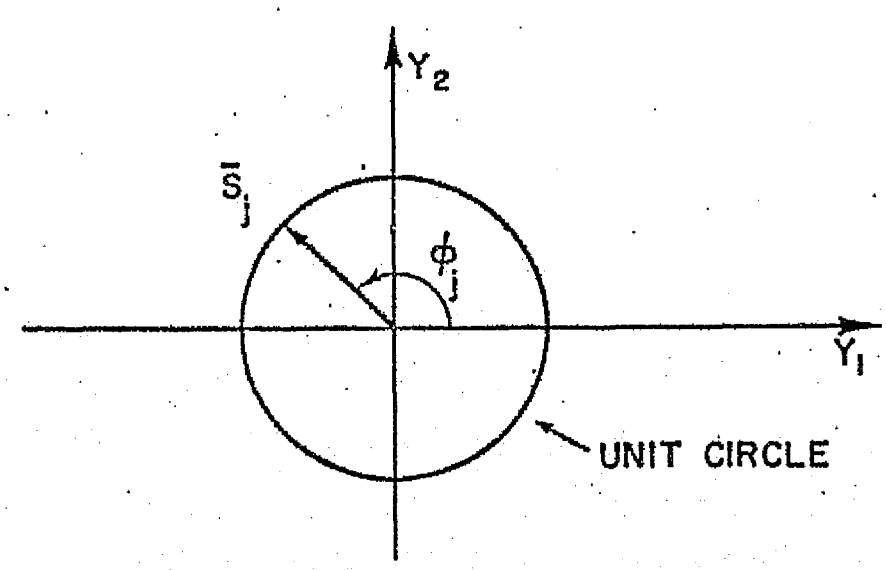
The $\overline{S}_j$ are unit vectors on the two-dimensional unit circle, as they should be to conform to the development in the previous chapter. The corresponding transmittable signal set is

$$V\cos(\omega_c t + \phi_j), \qquad 0 \le t \le T, \qquad j = 1, \ldots, M$$

and with this choice of basis functions, the phase angle ϕ_j corresponds to the angle that the signal vector $\overline{S}_j$ makes with the positive y_1 axis, where

$$\overline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda\,\overline{S}_j + \overline{N} \tag{3.4}$$

is the normalized received vector (see Figure 3.1). So, when D is restricted to two, the only allowed variation between the different transmittable signals is the reference phase angle.



Relating the Phase Angle in the Transmittable Signal to the Corresponding Signal Vector in Two Dimensions

FIGURE 3.1

3.1 Optimal Signal Selection in Two Dimensions

For a fixed set of $\{\bar{S}_j\}$ on the two-dimensional unit circle, and employing the optimum receiver for this particular signal set (namely, choosing that signal which corresponds to the maximum correlator output), from (2.31) the probability of detection is given by

$$P_D\left(\lambda; \{\bar{S}_i\}\right) = \frac{1}{M} \sum_{i=1}^{M} \Pr\left[(\bar{Y}\cdot\bar{S}_i) = \max_j \ (\bar{Y}\cdot\bar{S}_j) \Big/ \bar{S}_i \text{ was transmitted} \right]$$

When P_D is written in this manner there is no restriction on dimensionality. However, in two dimensions we can express P_D in the following way:

Theorem 3.1. For the received vector $\bar{Y}$, given by (3.4), where $\bar{N}$ has a Gaussian probability density function with covariance matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the probability of detection can be written

$$P_D\left(\lambda; \{\bar{S}_i\}\right) = \frac{1}{\pi} e^{-\lambda^2/2} \int_0^{\infty} r\, dr\, e^{-\frac{1}{2}r^2} \frac{1}{M} \sum_{i=1}^{M} \int_0^{\theta_i/2} e^{\lambda r \cos\alpha}\, d\alpha \tag{3.5}$$

where the angle θ_i is the angle between signal vectors $\bar{S}_i$ and $\bar{S}_{i+1}$ (numbering the signals consecutively around the unit circle and defining θ_M as the angle between $\bar{S}_M$ and $\bar{S}_1$).

Hence, we clearly have

$$\sum_{i=1}^{M} \theta_i = 2\pi \quad \text{and } \theta_i \geq 0, \ i = 1, \ldots, M. \tag{3.6}$$

Proof: Since the additive noise is independent of the a priori signal distribution, we can write

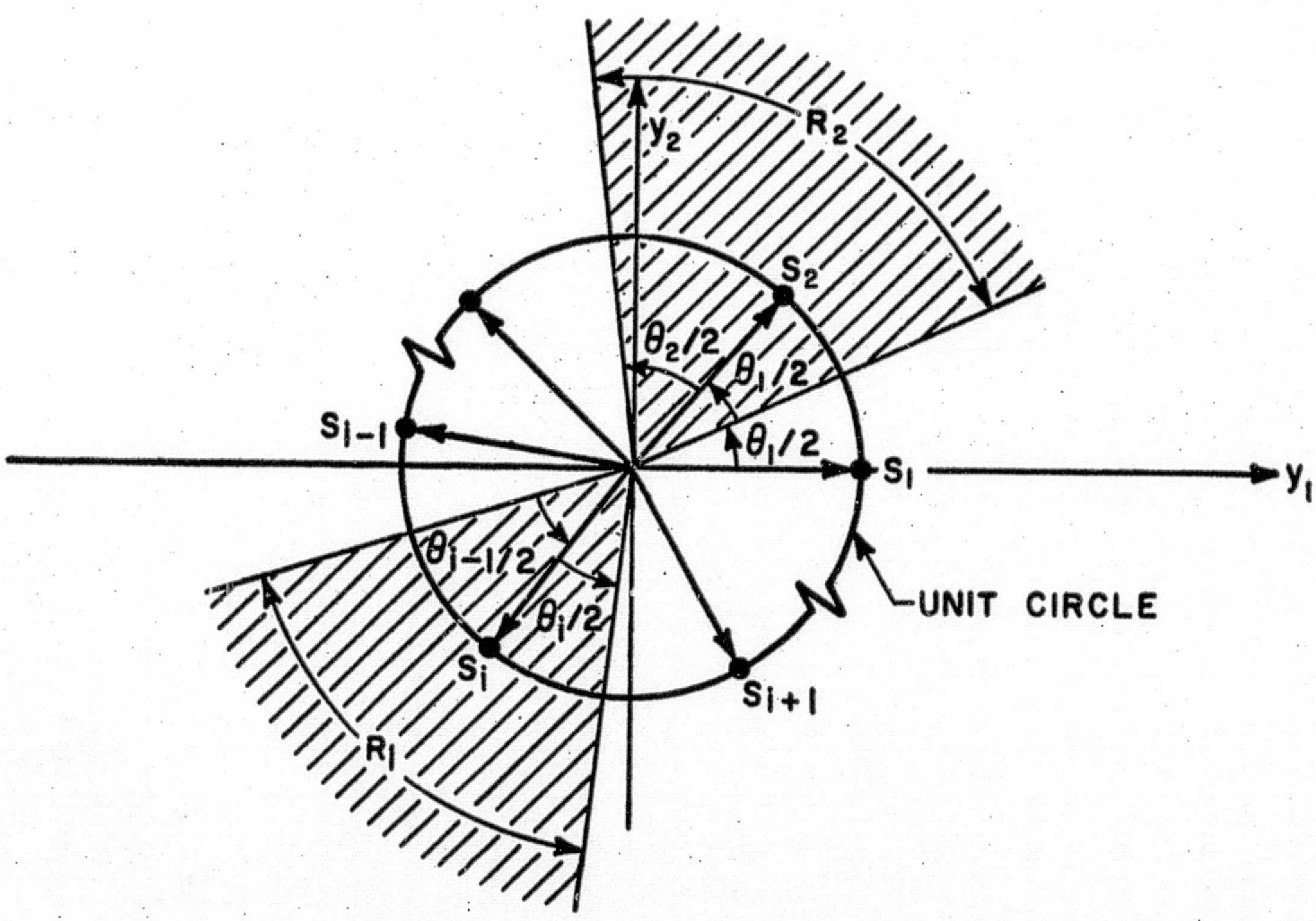
$$p(\bar{Y}/\bar{S}_i) = p_N(\bar{Y} - \lambda\bar{S}_i) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(\bar{Y} - \lambda\bar{S}_i)\cdot(\bar{Y} - \lambda\bar{S}_i)\right] \tag{3.7}$$

where $p_N(\ \ )$ is the noise p. d. f.

P_D can then be expressed as

$$P_D\left(\lambda\,\{\bar{S}_i\}\right) = \frac{1}{M}\sum_{i=1}^{M}\int_{R_i} p(\bar{Y}/\bar{S}_i)\, d\,|\bar{Y}| \tag{3.8}$$

where R_i is the region where $(Y\cdot S_i) \geq (Y\cdot S_j)$, $j = 1, \ldots, M$ (see Figure 3.2).



R_i = REGION WHERE $\bar{Y}\cdot\bar{S}_i \geq \bar{Y}\cdot\bar{S}_j$

$j = 1, \ldots, M$

FIGURE 3.2

Let

$$\bar{Y} = r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \tag{3.9}$$

and define

$$\Delta_1 = \frac{\theta_1}{2}$$

$$\Delta_2 = \theta_1 + \frac{\theta_2}{2}$$

$$\Delta_i = \sum_{j=1}^{i-1} \theta_j + \frac{\theta_i}{2} \tag{3.10}$$

Also

$$\bar{S}_i = \begin{pmatrix} s_1^i \\ \\ s_2^i \end{pmatrix} = \begin{pmatrix} \cos\left[\sum_{j=1}^{i-1} \theta_j\right] \\ \\ \sin\left[\sum_{j=1}^{i-1} \theta_j\right] \end{pmatrix} \tag{3.11}$$

Substitution yields

$$P_D\left(\lambda; \{\bar{S}_i\}\right) = \frac{1}{M\pi} e^{-\lambda^2/2} \int_o^\infty dr\ r e^{-\frac{1}{2} r^2}$$
$$\left(\sum_{i=1}^{M} \int_{\Delta_{i-1}}^{\Delta_i} \frac{1}{2} d\,\theta \exp\left[\lambda r\left\{\left(s_1^i\right) \cos\theta + \left(s_2^i\right) \sin\theta\right\}\right]\right) \tag{3.12}$$

which can be further reduced to

$$P_D\left(\lambda; \{\theta_i\}\right) = \frac{1}{M\pi} e^{-\lambda^2/2} \int_o^\infty dr\ r e^{-\frac{1}{2} r^2} I(\lambda r)$$

where

$$I(\lambda r) = \sum_{i=1}^{M} \int_o^{\theta_i/2} e^{\lambda r \cos\alpha} d\alpha \tag{3.13}$$

QED

Hence the probability of detection depends only on the SNR, λ, and the set of angles $\{\theta_i\}$ between the adjacent signal vectors. This indicates that P_D is independent of rotations of the signal vectors about the origin, which is equivalent to saying that P_D depends only on signal vector inner products and is independent of orthogonal transformations made on them. We prove this fact for all dimensions in the next chapter. P_D is also independent of the order in which the signals are placed on the unit circle, so long as the angle between $\tilde{S}_i$ and the signal vector adjacent to it in the counter-clockwise direction remains θ_i. This characteristic is true only when D=2, since for D>2, the concept of numbering the signals requires more criteria for ordering.

<u>Theorem 3.2.</u> In two dimensions, the optimal signal set (optimal in the sense of maximizing the probability of detection) consists of the M signal vectors equally spaced around the unit circle, and moreover, this is the optimum for all signal-to-noise ratio.

<u>Proof:</u> It is sufficient to prove that $I(\lambda r)$ is maximized at every $\lambda r > 0$ by choosing $\theta_i = \frac{2\pi}{M}$, $i = 1, \ldots, M$, (i.e., equal spacing). This is accomplished by showing that any other choice of $\{\theta_i\}$ will be less than this.

Let $I_o(\lambda r)$ correspond to the choice

$$\theta_i = \frac{2\pi}{M}, \; i = 1, \ldots, M. \tag{3.14}$$

Let

$$f(\beta) = \int_o^\beta e^{k \cos \alpha} \, d\alpha \tag{3.15}$$

$f(\beta)$ is a convex downward function of β, for $0 \leq \beta \leq \pi$ and any k. This is true because

$$\frac{\partial^2 F(\beta)}{\partial \beta^2} = (-k \sin \beta) \, e^{k \cos \beta}$$

is either always ≥ 0 or always ≤ 0 depending on the given value of k. In our case $k = \lambda r$ is always ≥ 0. Because of the convexity of $f(\beta)$ (see Figure 3.3),

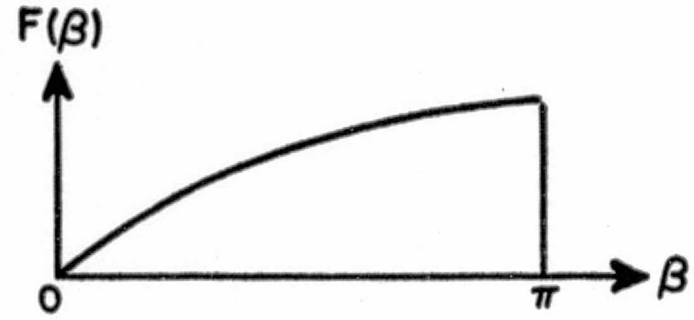


FIGURE 3.3

$$F\left(\sum_{i=1}^{M} \gamma_i \ \beta_i\right) \geq \sum_{i=1}^{M} \gamma_i \ F\left(\beta_i\right) \tag{3.16}$$

where

$$\sum_{i=1}^{M} \gamma_i = 1, \ \ \gamma_i \geq 0, \ i = 1, \ldots, M$$

and

$$\beta_i \ \epsilon [0, \ \pi], \qquad i = 1, \ldots, M.$$

Let

$$\gamma_i = \frac{1}{M}, \qquad \text{and} \qquad \beta_i = \frac{\theta_i}{2}$$

then

$$\sum_{i=1}^{M} \beta_i = \pi$$

Substituting

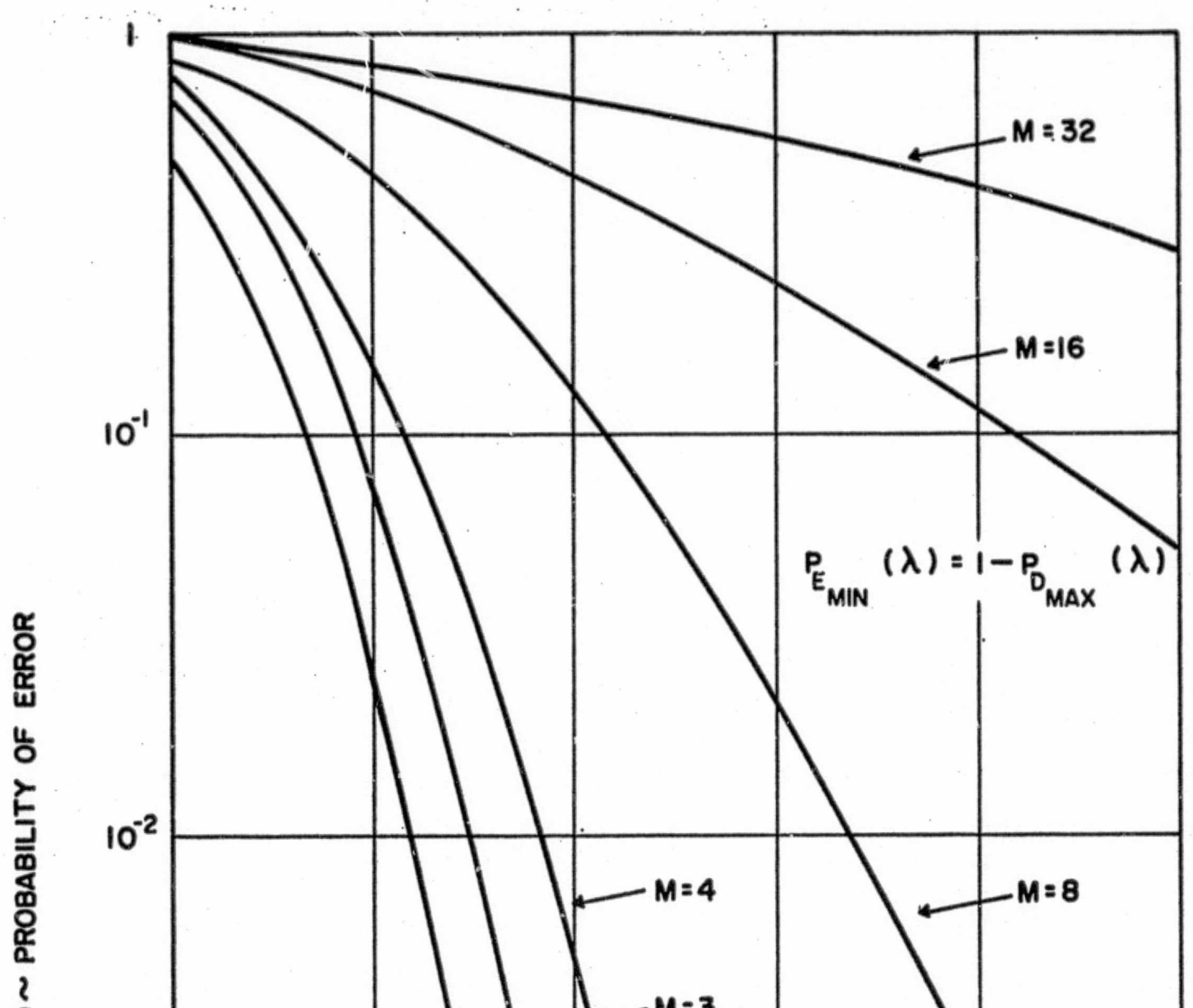
$$M \ F\left(\frac{\pi}{M}\right) \geq \sum_{i=1}^{M} F\left(\frac{\theta_i}{2}\right) \tag{3.17}$$

Thus

$$I_o(\lambda r) = \frac{1}{M\pi} \sum_{i=1}^{M} \int_o^{\pi/M} e^{\lambda r \cos \alpha} \ d\alpha \ \ \geq$$

$$\frac{1}{M\pi} \sum_{i=1}^{M} \int_o^{\theta_i/2} e^{\lambda r \cos \alpha} \ d\alpha = I(\lambda r) \tag{3.18}$$

<u>QED</u>



Probability of Error vs. Signal-to-Noise Ratio for the Optimal Signal Set When 0 = 2

FIGURE 3.4

The probability of detection for this optimal signal set is

$$P_D\left(\lambda;\left\{\theta_i=\frac{2\pi}{M}\right\}\right) = P_{D_{max}}(\lambda) = \frac{1}{\pi} e^{-\lambda^2/2} \int_0^{\infty} dr\, r\, e^{-\frac{1}{2} r^2} \int_0^{\frac{\pi}{M}} e^{\lambda r \cos \alpha} d\alpha \tag{3.19}$$

which can be rewritten as

$$P_{D_{max}}(\lambda) = 2\int_0^{\infty} G(y)\, dy \int_{\left[y \cot\frac{\pi}{M} - \lambda\right]}^{\infty} G(x)\, dx \tag{3.20}$$

where

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2}$$

The minimum probability of error $P_{E_{min}}(\lambda) = 1 - P_{D_{max}}(\lambda)$ is plotted in Figure 3. 4.

Note that the optimum receiver and optimal signal set were found under the assumption of an equi-likely a priori signal distribution. However, the probability of detection of the resulting optimum system is independent of the a priori distribution, which is an excellent property since the a priori distribution is not really known in advance in practical situations. This does not mean, however, that this system is optimum for a known non-equi-likely a priori distribution.

3.2 Communication Efficiency and Channel Capacity for Two Dimensional Signal Sets

Communication Efficiency is defined as

$$\beta = \frac{\text{Average Received Signal Power}}{[\Phi_c]\ [\text{Information Rate of Source in bits/sec.}]} \tag{3.21}$$

In the basic model, we have assumed that successive messages are statistically independent, and that they are equally likely. Hence,

$$\begin{aligned} H &= \text{information rate of the source} \\ &= \log_2 M(\text{bits/message}) \\ &= \frac{1}{T} \log_2 M(\text{bits/second}). \end{aligned} \tag{3.22}$$

From (2.6)

$$P_{av} = \frac{V^2}{2}$$

Therefore

$$\beta = \frac{T V^2}{2 \Phi_c \log_2 M} \tag{3.23}$$

and from (2.29)

$$\lambda^2 = \frac{V^2 T}{2 \Phi_c} = \beta \log_2 M = \beta T H \tag{3.24}$$

$P_{D_{max}}$ can then be expressed in terms of β and M as

$$P_{D_{max}} = 2 \int_0^\infty G(y)\, dy \int_{\left[\frac{y}{\tan \frac{\pi}{M}} - \sqrt{\beta \log_2 M}\right]}^\infty G(x)\, dx \tag{3.25}$$

The information rate at the output of an additive Gaussian continuous channel is (Reference 3.1)

$$R = \frac{1}{2} \int_{-\infty}^\infty \log_2 \left[\frac{P_N(f) + P_s(f)}{P_N(f)}\right] df$$

$$= \int_0^\infty \log_2 \left[1 + \frac{P_s(f)}{P_N(f)}\right] df \tag{3.26}$$

where $P_s(f)$ and $P_N(f)$ are the signal and noise spectral densities, respectively. If the channel is band-limited to W cps on either side of the carrier, and the noise has flat spectral density, the total noise power is $N = 4W \Phi_c$, and

$$R = 2 \int_0^W \log_2 \left[1 + \frac{P'_s(f)}{\Phi_c}\right] df$$

where

$$P'_s(f) = P_s(f + f_c)$$

Channel capacity is the maximum attainable value of R, under a given power restriction on the signal. If P_{av} is the maximum allowable signal power, R is maximized by a Gaussian signal that also has a flat spectral density, say S. Thus

$$P_{av} = 4WS$$

and

$$C_W = 2W \log_2 \left[1 + \frac{P_{av}}{4W\ \Phi_c} \right] \tag{3.27}$$

Since C_W is a monotonically increasing function in W, if there is no bandwidth restriction, we can allow $W \to \infty$, and

$$C_\infty = \frac{P_{av}}{2\ \Phi_c} \log_2 e > C_W \text{ for all W.} \tag{3.28}$$

By Shannon's Theorem, it is possible to transmit H bits/sec. with zero error if and only if $H < C$. Suppose H is fixed at some value $m = \frac{1}{T} \log_2 M$.

Hence

$$M = 2^{mT} = 2^{HT} \tag{3.29}$$

A method of obtaining arbitrarily small probability of error is to allow T to become sufficiently large. In the limit as $T \to \infty$, for fixed H, $M \to \infty$. Likewise $\lambda \to \infty$. β, however, remains constant. For a given coding scheme the probability of error is a function only of β and M.

When there is no restriction on bandwidth, Shannon's Theorem states

$$P_D \to 1 \quad \text{as } M \to \infty \quad \text{if } H < C_\infty$$

$$P_D \to 0 \quad \text{as } M \to \infty \quad \text{if } H > C_\infty \tag{3.30}$$

For $H < C_\infty$, we get

$$H < \frac{P_{av}}{2\ \Phi_c} \log_2 e \tag{3.31}$$

or

$$\beta \log_2 e > 2 \tag{3.32}$$

Thus as $M \rightarrow \infty$,

$$\begin{aligned} P_D &\rightarrow 1 \quad \text{for} \quad \beta \log_2 e > 2 \\ P_D &\rightarrow 0 \quad \text{for} \quad \beta \log_2 e < 2 \end{aligned} \tag{3.33}$$

So, for a given H and given channel, the minimum average signal power required for zero error is given by

$$P_{av_{min}} = 2\,\Phi_c H \log_e 2 \tag{3.34}$$

Consider now the application of these results to the case when $D = 2$. In (3.25), for large M

$$\frac{y}{\tan \frac{\pi}{M}} - \sqrt{\beta \log_2 M} \approx \frac{yM}{\pi} - \sqrt{\beta \log_2 M} \rightarrow \infty$$

as $M \rightarrow \infty$ for every $y > 0$. Thus

$$\lim_{M \rightarrow \infty} P_{D_{max}} = 0 \text{ for every } \beta. \tag{3.35}$$

Strictly speaking, the true bandwidth of M-phase modulation systems is undefinable because the process is non-stationary. (We are speaking here of a stochastic process. In the previous chapter the definition of bandwidth was for deterministic waveforms and employed the percent energy concept.) If a substitute bandwidth is defined as the ratio of number of degrees of freedom, D, to T, the message length, then

$$C_W = 2W \log_2 \left[1 + \frac{P_{av}}{2W\Phi_c} \right] \tag{3.36}$$

where now $W = \frac{D}{T}$. For D fixed at 2, as $T \rightarrow \infty$, $M \rightarrow \infty$ and

$$\lim_{T \rightarrow \infty} C_W = 0 \tag{3.37}$$

Since H is fixed at a value greater than zero, $H > C_{\infty} = 0$, and since

$$\lim_{T \rightarrow \infty} P_{D_{max}} = 0$$

from (3.35), Shannon's Theorem has been substantiated. T increasing and D fixed is an example of a band decreasing code. Regular simplex and orthogonal codes (to be discussed later) are examples of band increasing codes.

Even though signal design for finite time, T, is our primary purpose, it is significant to show that the optimal results do satisfy Shannon's limit theorems for large T.

3.3 Partial Ordering of the Class of Two Dimensional Signal Sets

In the previous chapter, the question of whether or not the optimal signal choice is dependent on the signal-to-noise ratio, λ, was raised and was said to be a significant part of the optimal signal design problem. In section 3.1, we prove that in the two-dimensional case, at any rate, the optimal signal set is indeed independent of signal-to-noise ratio. The question still remains, however, as to whether it is possible to have two suboptimal signal sets, say $\{\bar{S}_i, i=1, \ldots, M\}$ and $\{\bar{S}'_i, i=1, \ldots, M\}$ such that the probability of detection for $\{\bar{S}_i\}$, namely $P_D\left(\lambda; \{\bar{S}_i\}\right)$, is larger than that for $\{\bar{S}'_i\}$ for some λ while the reverse is true for other λ.

The problem involved here can be phrased in terms of a partial ordering of the class of signal sets (or equivalently a partial ordering of the class of non-negative definite M x M matricies of given rank). We induce a partial ordering by saying:

$\{\bar{S}_i\}$ is <u>preferred</u> to $\{\bar{S}'_i\}$ if and only if

$$P_D\left(\lambda; \{\bar{S}_i\}\right) > P_D\left(\lambda; \{\bar{S}'_i\}\right) \text{ for all } \lambda > 0. \tag{3.38}$$

As in section 3.1, let us represent a set of M signals in two-dimensions by the angles between adjacent points on the unit circle; thus a set of M signals can be specified by the column vector

$$\bar{\theta} = \begin{pmatrix} \theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \theta_M \end{pmatrix}$$

where

$$\sum_{i=1}^{M} \theta_i = 2\pi.$$

The induced partial ordering is definitely non-empty because the optimal choice, namely equal spacing, is better than any other choice for all λ. However, we can say more. Consider the following

Definition:

One set of M signals (θ') is said to be more equally spaced than another set of M signals (θ) if they can be related by

$$(\theta') = (P)(\theta) \tag{3.39}$$

where the M x M matrix P is such that

$$\sum_{i=1}^{M} p_{ij} = \sum_{j=1}^{M} p_{ij} = 1 \text{ for all } i, j = 1, \ldots, M$$

and

$$p_{ij} \geq 0 \text{ for all } i, j = 1, \ldots, M.$$

Then we can prove the following

Theorem: 3.3

Suppose (θ') is more equally spaced than (θ). Then,

$$P_D\left(\lambda; (\theta')\right) \geq P_D\left(\lambda; (\theta)\right) \text{ for all } \lambda \geq 0 .$$

Proof:

As always, assume all signals equi-likely.

Since

$$(\theta') = (P)(\theta)$$

$$\theta_i' = \sum_{j=1}^{M} p_{ij}\,\theta_j$$

Then, because of the convexity in β of $\int_0^{\beta} e^{k\cos\alpha}\,d\alpha$,

$$I\left(\lambda r; (\theta')\right) = \sum_{i=1}^{M} \int_0^{\theta_i'/2} e^{\lambda r\cos\alpha}\,d\alpha$$

$$= \sum_{i=1}^{M} \int_0^{\sum_{j=1}^{M} p_{ij}\,\theta_j/2} e^{\lambda r\cos\alpha}\,d\alpha \geq \sum_i \sum_j p_{ij} \int_0^{\theta_j/2} e^{\lambda r\cos\alpha}\,d\alpha$$

$$= \sum_{j=1}^{M} \int_0^{\theta_j/2} e^{\lambda r\cos\alpha}\,d\alpha = I\left(\lambda r, (\theta)\right) \tag{3.40}$$

Since this inequality holds for all $\lambda r \geq 0$

$$P_D\left(\lambda;(\theta')\right) \geq P_D\left(\lambda;(\theta)\right) \text{ for all } \lambda \geq 0$$

QED

3.4 Example Demonstrating Dependence of Some Suboptimal Signal Sets on SNR

Hence the subclass of signal sets which can be related as in the above definition is totally ordered. However, not all sets of signals can be so related. If one signal set has a higher probability of detection for some λ, it is not necessarily true that it has a higher P_D for all λ. The following example demonstrates both of these facts.

Theorem: 3.4

Let $(\theta') = \left(\theta_1' = \frac{\pi}{4};\ \theta_2' = \frac{\pi}{2};\ \theta_3' = \frac{5\pi}{4}\right)$

and $(\theta) = \left(\theta_1 = \frac{\pi}{8};\ \theta_2 = \frac{3\pi}{4};\ \theta_3 = \frac{9\pi}{8}\right)$

(Since θ_1 is less than θ'_1, θ'_2, and θ'_3, and since θ'_3 is greater than θ_1, θ_2, and θ_3, $(\bar{\theta}')$ and $(\bar{\theta})$ cannot be related by a matrix P having the characteristics indicated above).

Then

$$P_D\left(\lambda;(\theta)\right) > P_D\left(\lambda;(\theta')\right) \text{ for small } \lambda$$

and

$$P_D\left(\lambda;(\theta)\right) < P_D\left(\lambda;(\theta')\right) \text{ for large } \lambda.$$

Proof:

For small λ,

Differentiating (3.13) with respect to λ and setting $\lambda = 0$ yields

$$\left.\frac{\partial P_D\left(\lambda;(\theta)\right)}{\partial\lambda}\right|_{\lambda=0} = \frac{1}{3\pi}\sum_{1}^{3} \sin\frac{\theta_i}{2}$$

$$= \frac{1}{3\pi}\left[\sin\frac{\pi}{16} + \sin\frac{6\pi}{16} + \sin\frac{9\pi}{16}\right]$$

$$\approx \frac{1}{3\pi}\left[2.09976\right].$$

$$\left.\frac{\partial P_D\left(\lambda;(\theta')\right)}{\partial\lambda}\right|_{\lambda=0}$$

$$\approx \frac{1}{3\pi}\left[2.01367\right].$$

Therefore, in the neighborhood of $\lambda = 0$

$$P_D\left(\lambda;(\theta)\right) > P_D\left(\lambda;(\theta')\right).$$

For large λ,

we must show that for λ sufficiently large

$$\Delta(\lambda) = P_D\left(\lambda;\,(\theta')\right) - P_D\left(\lambda;(\theta)\right)$$

$$= \frac{1}{3\pi}\, e^{-\lambda^2/2} \int_0^\infty r\, dr\, e^{-\frac{1}{2}r^2}\,.$$

$$\left\{ \int_{\frac{\pi}{16}}^{\frac{2\pi}{16}} - \int_{\frac{4\pi}{16}}^{\frac{6\pi}{16}} + \int_{\frac{9\pi}{16}}^{\frac{10\pi}{16}} e^{\lambda r \cos\alpha}\, d\alpha \right\} > 0$$

The integral from $\frac{9\pi}{16}$ to $\frac{10\pi}{16}$ w.r.t.α is always positive.

Hence:

$$\Delta(\lambda) > \frac{1}{3\pi}\, e^{-\lambda^2/2} \int_0^\infty r\, dr\, e^{-\frac{1}{2}r^2} \left\{ \int_{\frac{\pi}{16}}^{\frac{2\pi}{16}} - \int_{\frac{4\pi}{16}}^{\frac{6\pi}{16}} d\alpha\, e^{\lambda r \cos\alpha} \right\}$$

for all $\lambda \geq 0$.

Since

$$\int_{\frac{\pi}{16}}^{\frac{2\pi}{16}} d\alpha\, e^{\lambda r \cos\alpha} \geq \frac{\pi}{16}\, e^{\lambda r \cos\frac{2\pi}{16}}$$

for all $\lambda r \geq 0$

and

$$\int_{\frac{4\pi}{16}}^{\frac{6\pi}{16}} d\alpha\, e^{\lambda r \cos\alpha} \leq \frac{2\pi}{16}\, e^{\lambda r \cos\frac{4\pi}{16}}$$

for all $\lambda r \geq 0$

it is sufficient to show that

$$\Delta(\lambda) > \left(\frac{1}{3}\right)\left(\frac{1}{16}\right) e^{-\lambda^2/2} \int_0^\infty r\, dr\, e^{-\frac{1}{2}r^2} \left\{ e^{\lambda r \cos\frac{2\pi}{16}} - 2\, e^{\lambda r \cos\frac{4\pi}{16}} \right\}$$

is greater than zero for λ sufficiently large.

Since:

$$\int_0^\infty r\,dr\, e^{-\frac{1}{2} r^2 + ar}$$

$$= 1 + a\sqrt{2\pi}\; e^{a^2/2} \left[1 - \text{erf}\,(a)\right],$$

where

$$\text{erfc}\,(a) = \int_a^\infty \frac{e^{-\frac{1}{2}t^2}\,dt}{\sqrt{2\pi}},$$

it is sufficient to show

$$\Delta(\lambda) > \left(\frac{1}{3}\right)\left(\frac{1}{16}\right) e^{-\lambda^2/2} \left\{ 1 + \lambda\left(\cos\frac{2\pi}{16}\right)\sqrt{2\pi}\; e^{\frac{\lambda^2\cos^2\frac{2\pi}{16}}{2}} \cdot \right.$$

$$\left. \left[1 - \text{erf}\left(\lambda\cos\frac{2\pi}{16}\right)\right] - 2\left[1 + \lambda\left(\cos\frac{4\pi}{16}\right)\sqrt{2\pi}\; e^{\frac{\lambda^2\cos^2\frac{4\pi}{16}}{2}} \left[1 - \text{erf}\left(\lambda\cos\frac{4\pi}{16}\right)\right]\right]\right\} > 0$$

or equivalently that

$$1 + \lambda\left(\cos\frac{2\pi}{16}\right)\sqrt{2\pi}\; e^{\frac{\lambda^2\cos^2\frac{2\pi}{16}}{2}} \left[1 - \text{erf}\left(\lambda\cos\frac{2\pi}{16}\right)\right]$$

$$- 2\left\{1 + \lambda\left(\cos\frac{4\pi}{16}\right)\sqrt{2\pi}\; e^{\frac{\lambda^2\cos^2\frac{4\pi}{16}}{2}} \left[1 - \text{erf}\left(\lambda\cos\frac{4\pi}{16}\right)\right]\right\} > 0$$

which implies

$$\left(\cos\frac{2\pi}{16}\right) e^{\frac{\lambda^2\cos^2\frac{2\pi}{16}}{2}} \left[1 - \text{erf}\left(\lambda\cos\frac{2\pi}{16}\right)\right]$$

$$- 2\left(\cos\frac{4\pi}{16}\right) e^{\frac{\lambda^2\cos^2\frac{4\pi}{16}}{2}} \left[1 - \text{erf}\left(\lambda\cos\frac{4\pi}{16}\right)\right] > \frac{1}{\lambda\sqrt{2\pi}}$$

Using the inequality

$$\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\left\{\frac{1}{x}-\frac{1}{x^3}\right\} < \operatorname{erf} x < \frac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}} \quad \text{for all } x > 0$$

(Reference 3.5) to make the substitutions

$$\operatorname{erf}\left(\lambda \cos\frac{2\pi}{16}\right) < \frac{e^{-\frac{1}{2}\lambda^2\cos^2\frac{2\pi}{16}}}{\lambda\cos\frac{2\pi}{16}\sqrt{2\pi}}$$

and

$$\operatorname{erf}\left(\lambda \cos\frac{4\pi}{16}\right) > \frac{e^{-\frac{1}{2}\lambda^2\cos^2\frac{4\pi}{16}}}{\sqrt{2\pi}}\left\{\frac{1}{\lambda\cos\frac{4\pi}{16}}-\frac{1}{\lambda^3\cos^3\frac{4\pi}{16}}\right\},$$

$\Delta(\lambda)$ can then be shown greater than zero for λ sufficiently large if we can show

$$\left(\cos\frac{2\pi}{16}\right) e^{\frac{\lambda^2\cos^2\frac{2\pi}{16}}{2}}\left[1-\frac{e^{-\frac{1}{2}\lambda^2\cos^2\frac{2\pi}{16}}}{\sqrt{2\pi}\,\lambda\cos\frac{2\pi}{16}}\right]$$

$$-2\left(\cos\frac{4\pi}{16}\right) e^{\frac{\lambda^2\cos^2\frac{4\pi}{16}}{2}}\left[1-\frac{e^{-\frac{1}{2}\lambda^2\cos^2\frac{4\pi}{16}}}{\sqrt{2\pi}}\left(\frac{1}{\lambda\cos\frac{4\pi}{16}}-\frac{1}{\lambda^3\cos^3\frac{4\pi}{16}}\right)\right] > \frac{1}{\lambda\sqrt{2\pi}}$$

which implies

$$\left(\cos\frac{2\pi}{16}\right) e^{\lambda^2\cos^2\frac{2\pi}{16}} > 2\left(\cos\frac{4\pi}{16}\right) e^{\lambda^2\cos^2\frac{4\pi}{16}} + \sqrt{\frac{2}{\pi}}\,\frac{1}{\lambda^3\cos^2\frac{4\pi}{16}}$$

But the last inequality holds whenever

$$\left(\cos\frac{2\pi}{16}\right) e^{\lambda^2\cos^2\frac{2\pi}{16}} > 2\left(\cos\frac{4\pi}{16}\right) e^{\lambda^2\cos^2\frac{4\pi}{16}} + 1 \text{ and } \lambda > 1\,,$$

which holds whenever

$$\left(\cos\frac{2\pi}{16}\right) e^{\lambda^2\cos^2\frac{2\pi}{16}} > 4\, e^{\lambda^2\cos^2\frac{4\pi}{16}} \text{ and } \lambda > 1$$

Since this last inequality is valid for all

$$\lambda^2 > \left[\frac{\ln 4 - \ln\left(\cos\frac{4\pi}{16}\right)}{\cos^2\frac{2\pi}{16} - \cos^2\frac{4\pi}{16}}\right] = \lambda_o^2 > 1$$

we can conclude

$$P_D\left(\lambda;(\theta)\right) < P_D\left(\lambda;(\theta')\right) \text{ for all } \lambda > \lambda_o$$

QED

Probabilities of error vs. signal-to-noise ratio are presented in Figure 3. 5 for (θ) and (θ'). The results show, for example, that

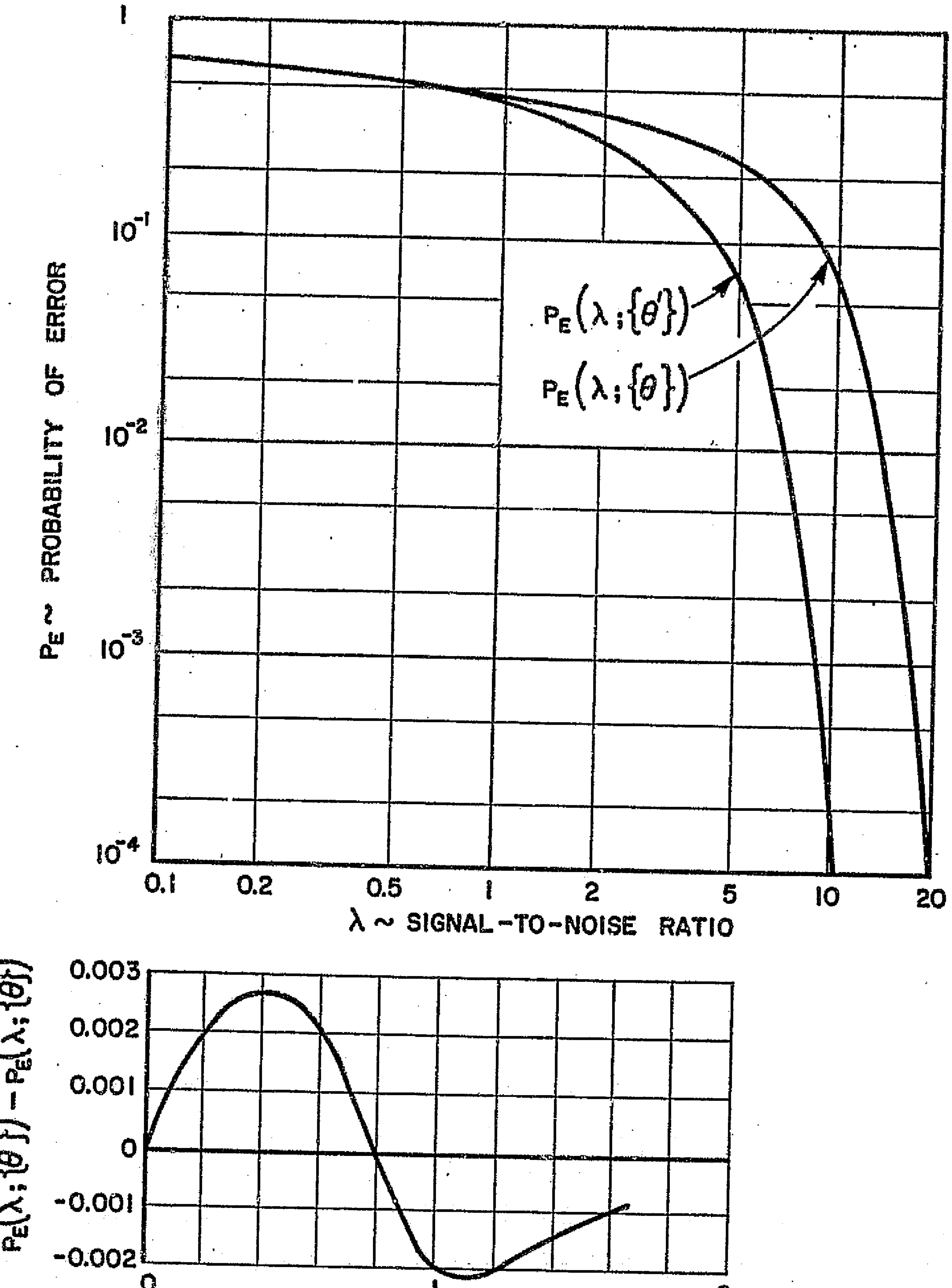
$$P_E\left(\lambda = 10; (\theta)\right) \approx 10^{-2}$$

$$P_E\left(\lambda = 10; (\theta')\right) \approx 10^{-4}$$

$$P_E\left(\lambda = 20; (\theta)\right) \approx 10^{-4}$$

$$P_E\left(\lambda = 20; (\theta')\right) \approx 10^{-14}$$

indicating that (θ') at $\lambda = 10$ is as good as (θ) at $\lambda = 20$.



PROBABILITY OF ERROR VS SNR FOR THE EXAMPLE IN THEOREM 3.4

FIGURE 3.5

REFERENCES

3.1 Fano, R.M., Transmission of Information, J. Wiley, 1961.

3.2 Shannon, C.E., "Probability of Error for Optimal Codes in Gaussian Channels", BSTJ, Vol. 38, No. 3, pp. 611-656, May, 1959.

3.3 Cahn, C.R., "Performance of Digital Phase Modulation Systems", IRE Trans. on Communication Systems, pp. 3-14, March, 1961, May, 1959.

3.4 Viterbi, A.J., "On Coded Phase Coherent Communications", IRE Trans. on Space Electronics and Telemetry, Vol. SET-7, No. 1, pp. 3-14, March, 1961.

3.5 Weber, C.L., "On Optimal Signal Selection for M-ary Alphabets with Two Degrees of Freedom", IEEE Trans. on Information Theory, April, 1965.

3.6 Feller, W., An Introduction to Probability Theory and Its Applications, J. Wiley, 1950.

3.7 Scholtz, R.A., and Reed, I.S., "N-Orthogonal Phase Modulated Codes", First IEEE Annual Communications Convention, Boulder, Colorado, June, 1965.

3.8 Viterbi, A.J., "On a Class of Polyphase Codes for the Coherent Gaussian Channel", IEEE International Convention, New York, N.Y., March, 1965.

IV. GENERAL THEORY

Let us begin by again giving a precise statement of the signal design problem, stated in the way that we will henceforth consider it.

Let Y be a D-dimensional vector random variable (real valued) of the form:

$$Y = \lambda S_j + Z \tag{4.1}$$

where Z is a Gaussian random vector with zero mean and whose covariance matrix is the D-by-D identity matrix, where S_j is one of M equally likely signal vectors, each S_j being a known unit vector in E_D (D-dimensional Euclidean Space) with $D \leq M$, and where $\lambda > 0$ is the signal-to-noise ratio. After observing Y we are asked to optimally determine (optimum in the sense of maximizing the probability of detection) which S_j has been transmitted. For a fixed signal set $\{S_j, j=1, \ldots, M\}$, it is well known that the probability of detection is maximized by the matched filter, which forms

$$E_i = (Y \cdot S_i), \; i = 1, \ldots, M$$

and decides S_j was transmitted if

$$E_j = \max_i E_i .$$

The corresponding probability of detection is

$$P_D\left(\lambda ; \{S_j\}\right) = \frac{1}{M} \sum_{j=1}^{M} \Pr\left(E_j = \max E_i \Big/ Y = \lambda S_j + Z\right) \tag{4.2}$$

The optimal signal design problem is to find that set of vectors $\{S_j\}$ which makes this probability a maximum for various M and D, and to determine the optimal signal set's dependence or independence of signal-to-noise ratio.

In this chapter we shall discuss some of the significant properties of the class of admissible signal sets (admissible in the sense that they satisfy all of the designated restrictions), and find certain subclasses of signal sets which contain the optimal sets. This will give certain characteristics which the optimal sets must possess and simultaneously reduce the size of the class which contains the optimal sets. Then in later chapters we will show that certain signal sets are optimum under different dimensionality restrictions, and indicate precisely in what sense they are optimum.

It will be convenient to denote the set of signal vector inner products, $\{\lambda_{ij}\}$, by the symmetric M-by-M matrix

$$\alpha = \begin{pmatrix} 1 & & \lambda_{ij} \\ & \ddots & \\ \lambda_{ji} & & 1 \end{pmatrix} \tag{4.3}$$

<u>LEMMA</u> 4.1 α is non-negative definite.

<u>Proof</u>

Define $S = (S_1, S_2, \ldots, S_M)$, a row of column vectors which is D-by-M.

Then $\alpha = S^* S$ and is M-by-M.

For any column vector a

$$a^* \alpha\, a = a^* S^* S\, a = (S\, a)^* (S\, a) \geq 0$$

since the last quantity is a sum of squares. Hence α is non-negative definite.

<u>QED</u>

Note that the rank of the matrix α is equal to the allowed degrees of freedom of the signal set. Thus α is M-by-M and has rank D.

Also, $P_D\left(\lambda ; \{S_j\}\right)$ is a non-decreasing function in λ with

$$P_D\left(0;\{S_j\}\right) = \frac{1}{M} \text{ for any set of } \{S_j\} \text{ and}$$

$$\lim_{\lambda\to\infty} P_D\left(\lambda;\{S_j\}\right) = 1 \text{ if the } S_j \text{ are all different.}$$

From (4. 2), we can write

$$\Pr\left(E_j = \max_i E_i / Y = \lambda S_j + Z\right)$$
$$= \int_{\Lambda_j} \frac{1}{(2\pi)^{D/2}} \exp\left[-\tfrac{1}{2}(Y-\lambda S_j)\cdot(Y-\lambda S_j)\right] d\,|Y| \tag{4.4}$$

where

Λ_j is the region where $(Y\cdot S_j) \geq (Y\cdot S_i)$ for $i \neq j$.

The integrand in (4. 4) is

$$e^{-\frac{1}{2}(Y-\lambda S_j)\cdot(Y-\lambda S_j)} = e^{-\frac{1}{2}\lambda^2}\, e^{-\frac{1}{2}(Y\cdot\gamma) + \lambda(Y\cdot S_j)} \tag{4.5}$$

Substituting these into (4. 2), we obtain

$$P_D\left(\lambda;\{S_j\}\right) = \frac{1}{M} e^{-\frac{1}{2}\lambda^2} \frac{1}{(2\pi)^{D/2}} \sum_{j=1}^{M} \int_{\Lambda_j}\cdots\int e^{-\frac{1}{2}(Y\cdot Y) + \lambda(Y\cdot S_j)}\, d\,|Y|$$

$$= \frac{1}{M} e^{-\frac{1}{2}\lambda^2} \frac{1}{(2\pi)^{D/2}} \int_{E_D}\cdots\int e^{-\frac{1}{2}(Y\cdot Y) + \lambda \max_j (Y\cdot S_j)}\, d|Y| \tag{4.6}$$

where E_D is the entire D-dimensional space. This can be expressed as

$$P_D\left(\lambda;\{S_j\}\right) = \frac{1}{M} e^{-\frac{1}{2}\lambda^2} \left\{E \quad e^{\lambda \max_j (Y\cdot S_j)}\right\} \tag{4.7}$$

where Y is now a D-variate Gaussian with zero mean and covariance matrix equal to the D-by-D identity matrix. When interpreted in this way, Y is independent of S_j. We shall adopt the notation

$$G(Y;\, m;\, C) \tag{4.8}$$

for a Gaussian variate, Y, with mean m, and covariance matrix C. Now let

$$\xi_j = (Y \cdot S_j), \quad j = 1, \ldots, M \tag{4.9}$$

and

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_M \end{pmatrix} \tag{4.10}$$

Then $\qquad \xi = S^* Y,$

$$E(\xi) = E(S^* Y) = S^* E(Y) = (0),$$

and

$$\mathrm{cov}(\xi) = E(\xi \xi^*) = E(S^* Y Y^* S) = S^* S = \alpha$$

Substituting:

$$P_D\left(\lambda; \{S_j\}\right) = \frac{1}{M} e^{-\frac{1}{2}\lambda^2} E\left\{e^{\lambda \max_j (Y \cdot S_j)}\right\}$$

$$= \frac{1}{M} e^{-\frac{1}{2}\lambda^2} E\left\{e^{\lambda \max_j \xi_j}\right\} = P_D(\lambda; \alpha) \tag{4.11}$$

where ξ has p.d.f. which is $G(\xi; 0; \alpha)$.

Hence, P_D is a function only of λ and the matrix of signal vector inner products, α, and is therefore invariant to any orthogonal transformation on the signal vectors. Thus it is sufficient to specify a signal set by its set of inner products.

Also, if we define

$$\phi(\lambda;\alpha) = E\left\{e^{\lambda \max_i \xi_i}\right\} \tag{4.12}$$

then

$$P_D(\lambda;\alpha) = \frac{1}{M} e^{-\frac{1}{2}\lambda^2} \phi(\lambda;\alpha) \tag{4.13}$$

and the optimization problem has been reduced to finding that α matrix which maximizes $\phi(\lambda;\alpha)$ in (4.12) for various λ.

We now define the class of admissible α, Γ, as those M-by-M symmetric non-negative definite matricies with one's along the main diagonal and all off diagonal elements ≤ 1 in magnitude.

The remainder of this chapter is spent in finding certain subclasses of Γ which contain the optimum α.

4.1 Convex Body Considerations; Small Signal-to-Noise Ratio

The maximization of $P_D(\lambda;\alpha)$ in the neighborhood of $\lambda = 0$ can be put in the context of Convex Body Theory. (For the necessary theory of convex bodies, see References 4.4 and 4.5.) To do this, let

$$H_S(Y) = \max_i (Y \cdot S_i) \tag{4.14}$$

$H_S(Y)$ is the support function of the polytope formed by the set of vectors $\{S_i\}$. The polytope in this case is the set

$$\left\{ Y \,\middle|\, Y = \sum_{i=1}^{M} \gamma_i S_i ; \sum_{i=1}^{M} \gamma_i = 1;\ \gamma_i \geqq 0,\ i = 1, \ldots, M \right\}$$

which is the convex hull generated by the $\{S_i\}$.

Substituting into (4.6)

$$P_D(\lambda;\alpha) = \frac{1}{M} e^{-\frac{1}{2}\lambda^2} \int_{E_D}\!\!\cdots\!\int \exp\left[\lambda H_S(Y) - \frac{(Y \cdot Y)}{2}\right] d|Y| \tag{4.15}$$

By noting that for any function f( )

$$\int_{E_D}\!\!\cdots\!\int f(Y)\, d|Y| = \int_0^{\infty} r^{D-1}\, dr \int_{\Omega_D} f(Y)\, d\Omega$$

where $r^2 = \sum_{i=1}^{D} y_i^2$, Ω_D is the surface of the D-dimensional unit sphere, and $d\Omega$ is the surface element on Ω_D, we can write $\phi(\lambda;\alpha)$ as

$$\phi(\lambda;\alpha) = \frac{1}{(2\pi)^{D/2}} \int_0^\infty r^{D-1}\, dr\, e^{-\frac{1}{2}r^2} \int_{\Omega_D} e^{\lambda r H_S(Y)}\, d\Omega \tag{4.16}$$

where Y is now of unit magnitude.

Since $\phi(0;\alpha) = 1$ for every α, if there is a choice of α which maximizes $\phi(\lambda;\alpha)$ for small λ (or if there is a choice of α which maximizes $\phi(\lambda;\alpha)$ independent of λ) it necessarily must maximize the derivative of $\phi(\lambda;\alpha)$ with respect to λ at the origin. Thus from (4.16)

$$\left.\frac{\partial\phi(\lambda;\alpha)}{\partial\lambda}\right|_{\lambda=0} = \frac{B}{(2\pi)^{D/2}} \cdot \int_0^\infty r^D\, dr\, e^{-\frac{1}{2}r^2} = \sqrt{2}\, \frac{\Gamma\left(\frac{D+1}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} \bar{B} \tag{4.17}$$

where

$$B = \int_{\Omega_D} H_S(\gamma)\, d\Omega, \tag{4.18}$$

$\bar{B}$ is the mean width of the convex body, which is equal to

$$\bar{B} = \frac{B}{\omega_D} \tag{4.19}$$

and where ω_D is the surface area of the D-dimensional unit sphere.

i.e., $\omega_D = \int_{\Omega_D} d\Omega$

or

$$\omega_D = \frac{2(\sqrt{\pi})^D}{\Gamma\left(\frac{D}{2}\right)}$$

$\Gamma(\cdot)$ in (4.17) and (4.19) is the gamma function.

Thus for a given D and M, a necessary condition for the optimum α is that it maximize the mean width. This formulation is independent of coordinate rotations. As defined here, the mean width is an average radial distance, averaged uniformly over the D-dimensional unit sphere. It is not an average diameter, as the name might imply.

Using the theory of convex bodies, it can be proven that when $D=M-1$, the polytope which maximizes the mean width is the regular simplex, which consists of M vectors having an inner product structure given by

$$\alpha_R = \begin{pmatrix} 1 & & \frac{-1}{M-1} \\ & \ddots & \\ \frac{-1}{M-1} & & 1 \end{pmatrix} \tag{4.20}$$

i.e., $\lambda_{ij} = \frac{-1}{M-1}$ for all $i \neq j$. From further geometrical considerations it can be shown that α_R is the only polytope which maximizes the mean width. Therefore for small λ (small in the sense that a first order approximation is sufficient), and $D=M-1$, the regular simplex is the optimum signal set. The maximum mean width for M points and $D<M-1$ will be less than that of the regular simplex. In the next section, we show that the mean width is not increased by allowing D to equal M. Thus, if D is left unspecified, the dimensionality in which the largest mean width is attained is $D=M-1$, and the corresponding signal set is the regular simplex.

EXAMPLE:

To agree with the two dimensional results of the previous chapter, we must be able to show the following:

THEOREM 4.1. The mean width for $D=2$ and M = any integer is maximized by equally spacing the M points on the unit circle.

Proof: First we show that in two dimensions the

$$\text{Mean Width} = \bar{B}_2 = \frac{1}{2\pi}\,(\text{Perimeter}). \tag{4.21}$$

Given any M points on the unit circle with corresponding angles $\{\theta_i\}$ as indicated in Figure 4.1, then the

$$\text{Perimeter} = P = \sum_{i=1}^{M} p_i \text{ where } p_i = 2\sin\frac{\theta_i}{2}$$

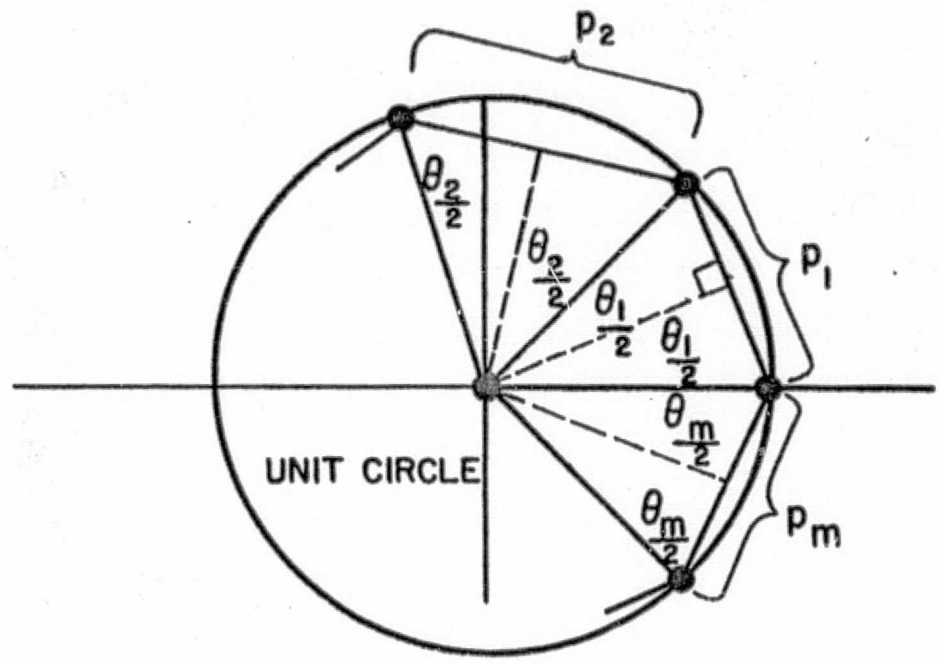


FIGURE 4.1

Hence

$$P = 2 \sum_{i=1}^{M} \sin \frac{\theta_i}{2} \tag{4.22}$$

$\bar{B}_2$ can be written

$$\bar{B}_2 = \frac{1}{2\pi} \int_0^{2\pi} L(\xi)\, d\xi \tag{4.23}$$

where, for any angle ξ, $L(\xi)$ is the radial distance at that angle, from the origin to the point where lines drawn perpendicular to the radial line first intersect the perimeter of the polygon. This is illustrated in Figure 4.2.

Now, from Figures 4.1 and 4.2

$$\bar{B}_2 = \frac{2}{2\pi} \sum_{i=1}^{M} \int_0^{\theta_i/2} (\cos \xi)\, d\xi = 2\, \frac{1}{2\pi} \sum_{i=1}^{M} \sin \left(\frac{\theta_i}{2} \right) = \frac{P}{2\pi} \tag{4.24}$$

Thus

$$\bar{B}_2 = \frac{1}{2\pi} \text{(Perimeter)}$$

for any arbitrary spacing of M points.

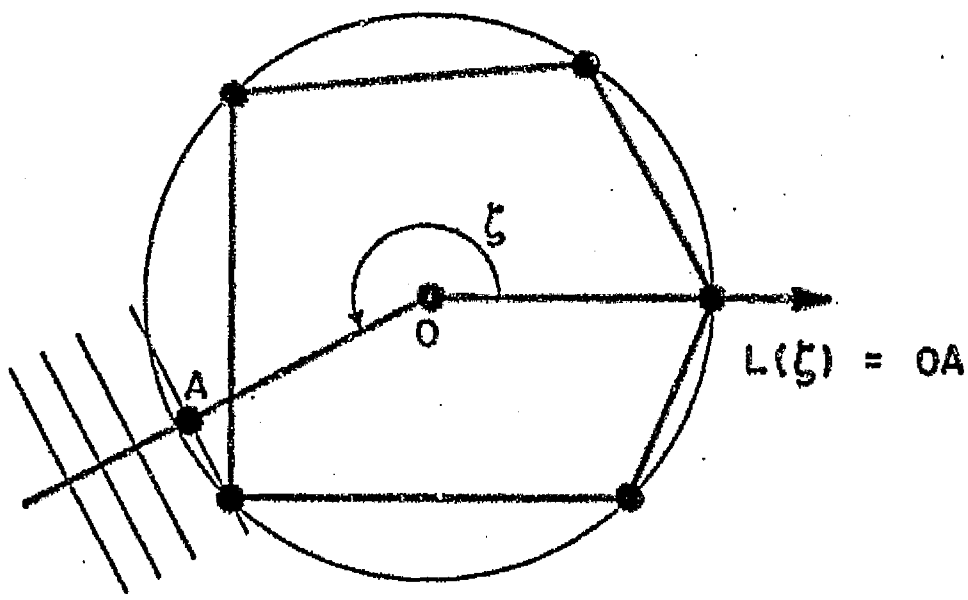


FIGURE 4.2

Now we maximize the perimeter. The function

$$\sin\left(\frac{\theta}{2}\right) \quad 0 \le \theta \le 2\pi$$

is convex downward. Because of its convexity, we can write

$$\sin\left(\sum_{i=1}^{M} \alpha_i x_i\right) \ge \sum_{i=1}^{M} (\sin x_i)\, \alpha_i$$

where

$$\sum_{i=1}^{M} \alpha_i = 1; \; \alpha_i \ge 0;$$

and

$$x_i \in [0, \pi]$$

Let

$$x_i = \frac{\theta_i}{2}$$

therefore

$$x_i \in [0, \pi] \text{ for all i.}$$

Let

$$\alpha_i = \frac{1}{M} \text{ . Thus } \alpha_i > 0$$

and

$$\sum_{i=1}^{M} \alpha_i = 1$$

Substituting:

$$\sin\left(\sum_{i=1}^{M} \frac{1}{M} \frac{\theta_i}{2}\right) = \sin\left(\frac{\pi}{M}\right) \geq \sum_{i=1}^{M} \frac{1}{M} \sin\left(\frac{\theta_i}{2}\right)$$

since

$$\sum_{i=1}^{M} \frac{1}{M} \frac{\theta_i}{2} = \frac{\pi}{M}$$

Thus

$$2M \sin\left(\frac{\pi}{M}\right) \geq \sum_{i=1}^{M} 2 \sin\left(\frac{\theta_i}{2}\right) = P$$

This proves that the perimeter is reduced if the θ' s are chosen different from $\theta_i = \frac{2\pi}{M}$, $i=1, \ldots, M$.

QED

In this section we have shown that finding optimal α for small λ is equivalent to maximizing the mean width. It is worth noting that the optimization can be phrased in the reverse order, namely, for a given M and D maximizing the mean width is equivalent to maximizing $P_D(\lambda; \alpha)$ for small λ. This is particularly significant because the geometric problem of maximizing the mean width for arbitrary M and D is in general still open.

It should be emphasized that if it were known that the optimum α were independent of signal-to-noise ratio for all M and D, then the problem would be reduced to maximizing the mean width for various M and D, and the optimal signal design problem would be strictly a geometric one. However, to date, the optimum α has been shown to be independent of λ only for the case when D=2. Also, the counter example in the previous chapter, indicating that the preference of suboptimal signal sets can indeed depend on λ, puts all the more emphasis on being able to demonstrate the optimal sets' dependence or lack of dependence on λ. However, all of the local optimal results that exist at present are independent of λ. The later chapters will discuss these results in detail.

We now find other characteristics which the optimal α must possess, thereby further reducing the size of the class containing the optimal sets.

4.2 Linearly Dependent vs. Linearly Independent Signal Sets

THEOREM 4.2. For each set of linearly independent vectors $\{S_i, i=1, \ldots, M\}$, there exists a set of linearly dependent vectors $\{S'_i, i=1, \ldots, M\}$ with a greater probability of detection at all signal-to-noise ratio.

Remark: This is not to say that all dependent sets are preferred over all independent sets. This theorem does say, however, that all optimal signal sets lie in the class of linearly dependent vectors. Or, it is sufficient to prove that P_D for the optimum α, namely $P_D(\lambda; \bar{\alpha})$ is greater than $P_D(\lambda;\alpha)$ for α corresponding to linearly dependent vectors.

Proof:

Since the dimensionality of the linearly independent signal set is M, there exists an M-1 flat through the tips of the $\{S_i\}$, defined by the M equations (see Figure 4.3);

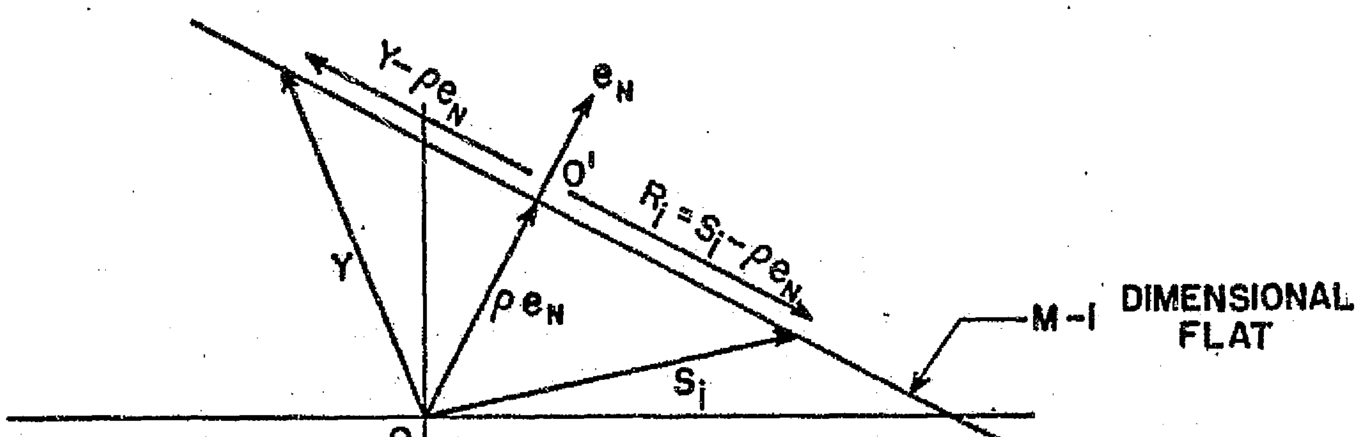


FIGURE 4.3

$$\left((S_i - \rho\, e_N) \cdot e_N \right) = 0, \qquad i = 1, \ldots, M$$

where e_N is the unit normal to the M-1 flat and ρ is the projected distance from O to the flat, which is the distance OO' in Figure 4.3. These defining equations can be rewritten

$$(S_i \cdot e_N) = \rho, \qquad i = 1, \ldots, M.$$

where $\rho \neq 0$. (If $\rho = 0$, this implies $(S_i \cdot e_N) = 0$, which implies each S_i is orthogonal to e_N, implying that the $\{S_i\}$ occupy only M-1 dimensions and are therefore linearly dependent.)

Let R_i be such that

$$S_i = R_i + \rho\, e_N;$$

then

$$\|S_i\| = \sqrt{(S_i \cdot S_i)} = 1 = \sqrt{(R_i \cdot R_i) + \rho^2}$$

and

$$\|R_i\| = \sqrt{1 - \rho^2} < 1$$

Now let

$$S'_i = \frac{R_i}{\sqrt{1 - \rho^2}}, \qquad i = 1, \ldots, M.$$

The S'_i are M unit vectors in the M-1 flat, hence they are linearly dependent.

As in (4.9), let

$$\xi'_i = (Y \cdot S'_i) = \frac{\left(Y \cdot R_i\right)}{\sqrt{1 - \rho^2}}$$

then

$$E(\xi'_i\, \xi'_j) = \frac{\left(R_i \cdot R_j\right)}{1 - \rho^2}$$

and

$$\xi_i = (Y \cdot S_i) = \sqrt{1-\rho^2}\left(\frac{(Y \,.\, R_i)}{\sqrt{1-\rho^2}}\right) + \rho\,(Y \cdot e_N)$$

$$= \sqrt{1-\rho^2}\;\xi'_i + \rho(\gamma \cdot e_N)$$

Therefore

$$\phi(\lambda;\alpha) = E\left\{e^{\lambda \max\limits_i \xi_i}\right\} = E\left\{\exp\left[\lambda \max_i \left(\sqrt{1-\rho^2}\;\xi'_i + \rho\,(Y \cdot e_N)\right)\right]\right\}$$

$$= E\left\{e^{\lambda\rho(Y \cdot e_N)}\; e^{\lambda\sqrt{1-\rho^2}\;\max\limits_i \xi'_i}\right\}$$

Let

$$u = (Y \cdot e_N)$$

The vector Y is a Gaussian variate, hence u is a Gaussian random variable with

$$E(u) = 0$$

$$E(u^2) = E(e_N^* YY^* e_N) = e_N^* e_N = 1$$

and

$$E(u\,\xi'_i) = E\left(e_N^* Y \frac{Y^* R_i}{\sqrt{1-\rho^2}}\right) = \frac{e_N^* R_i}{\sqrt{1-\rho^2}} = 0 \qquad i = 1, \ldots, M.$$

Thus u is independent of the ξ'_i.

Since

$$E\left\{e^{\lambda\rho\mu}\right\} = e^{\lambda^2\rho^2/2}$$

then

$$\phi(\lambda;\alpha) = e^{\lambda^2\rho^2/2}\; E\left(e^{\lambda\sqrt{1-\rho^2}\;\max\limits_i \xi'_i}\right)$$

and

$$P_D(\lambda;\alpha) = \frac{1}{M}\, e^{-\frac{\lambda^2}{2}(1-\rho^2)}\; E\left(e^{\lambda\sqrt{1-\rho^2}\;\max\limits_i \xi'_i}\right)$$

Define

$$P_D(\lambda;\alpha') = \frac{1}{M} e^{-\lambda^2/2} \, E\left(e^{\lambda \max_i \xi'_i}\right)$$

where

$$\alpha' = \begin{pmatrix} 1 & & \lambda'_{ij} \\ & \ddots & \\ \lambda'_{ji} & & 1 \end{pmatrix}$$

with

$$\lambda'_{ij} = (S'_i \cdot S'_j)$$

is the inner product matrix of the dependent vectors.

Therefore,

$$P_D\left(\lambda\sqrt{1-\rho^2};\alpha'\right) = P_D(\lambda;\alpha) \tag{4.25}$$

Since $\sqrt{1-\rho^2} < 1$ and P_D is a monotonically increasing function in λ, we have

$$P_D(\lambda;\alpha') > P_D(\lambda;\alpha) \text{ for all } \lambda. \tag{4.26}$$

QED

Stated another way, (4.24) indicates that the SNR required to attain a given error rate by the linearly dependent signal set is not as great as that required for the linearly independent signal set. We now use this result to get a precise comparison between certain signal sets as demonstrated in the following two important examples.

Example 1. Orthogonal Signal Set vs. the Regular Simplex.

The orthogonal signal set is characterized by an inner product matrix equal to the M-by-M identity matrix, which we shall denote by α_O, and that for the regular simplex, α_R, is given by (4.20). The orthogonal signal set is a linearly independent signal set. We assume the vectors of this set align themselves with the coordinate axes in E_M. If they do not, we can use an orthogonal transformation to align them. This does not alter P_D, since P_D is invariant to orthogonal transformations on the signal set.

To find ρ for α_o, we follow the same procedure as in the previous proof, i.e.,

$$(S_i \cdot e_N) = \rho, \ i = 1, \ldots, M.$$

Define

$$\bar{\rho} = \begin{pmatrix} \rho \\ \cdot \\ \cdot \\ \cdot \\ \rho \end{pmatrix}$$

Then

$$S^* e_N = \bar{\rho}$$

where as before $S = (S_1, \ldots, S_M)$.

Now

$$\bar{\rho}^* \bar{\rho} = M \rho^2 = (S^* e_N)^* (S^* e_N) = e_N^* e_N = 1$$

and

$$\rho^2 = \frac{1}{M} ,$$

As before

$$(S_i \cdot S_j) = \delta_{ij} = (R_i \cdot R_j) + \rho^2.$$

Thus

$$(R_i \cdot R_j) = \begin{cases} 1 - \rho^2 & \text{if} \quad i = j \\ -\rho^2 & \text{if} \quad i \neq j \end{cases}$$

and

$$(S'_i \cdot S'_j) = \begin{cases} 1 & \text{if} \quad i = j \\ \dfrac{-\rho^2}{1-\rho^2} = \dfrac{-1}{M-1} & \text{if} \quad i \neq j \end{cases}$$

which corresponds to the regular simplex signal structure, where

$$\alpha_R = \begin{pmatrix} 1 & & \frac{-1}{M-1} \\ & \ddots & \\ \frac{-1}{M-1} & & \ddots \\ & & & 1 \end{pmatrix}$$

By substitution, we have

$$P_D(\lambda;\alpha_o) = P_D\left(\lambda\sqrt{\frac{M-1}{M}};\alpha_R\right)$$

or $\left(\text{rescaling } \lambda \text{ i. e., } \lambda\sqrt{\frac{M-1}{M}} \rightarrow \lambda\right)$

$$P_D\left(\lambda\sqrt{\frac{M}{M-1}};\alpha_o\right) = P_D(\lambda;\alpha_R). \qquad (4.27)$$

Note that for small M, the improvement of α_R over α_o is greater than for large M. As $M\rightarrow\infty$, $\alpha_R \rightarrow \alpha_o$, and the signal sets become identical.

In the orthogonal signal set, the ξ_i are all independent Gaussian random variables with mean zero and unit variance. Hence $P_D(\lambda;\alpha_o)$ can be written directly as

$$P_D(\lambda;\alpha_o) = \frac{1}{M} e^{-\lambda^2/2} E\left\{e^{\lambda \max_i \xi_i}\right\} = \frac{1}{M} e^{-\lambda^2/2} \sum_{j=1}^{M} \int_{-\infty}^{\infty} e^{\lambda\xi_j} \frac{e^{-\frac{1}{2}\xi_j^2}}{\sqrt{2\pi}} d\xi_j$$

$$\underbrace{\int_{-\infty}^{\xi_j}\cdots\int}_{M-1 \text{ fold}} \frac{\exp\left[-\frac{1}{2}\sum_{i\neq j}\xi_i^2\right]}{(\sqrt{2\pi})^{M-1}} d\xi_1 \ldots d\xi_{j-1}\, d\xi_{j+1} \ldots d\xi_M = \int_{-\infty}^{\infty} G(x-\lambda)\, dx \left[\phi(x)\right]^{M-1}$$

(4.28)

where

$$\phi(x) = \int_{-\infty}^{x} G(y)\, dy \tag{4.29}$$

and

$$G(x) = \frac{1}{\sqrt{2\pi}} \quad e^{-\frac{1}{2} x^2} \tag{4.30}$$

By using (4.27), we also have a direct way of evaluating $P_D(\lambda; \alpha_R)$, namely

$$P_D(\lambda;\alpha_R) = \int_{-\infty}^{\infty} G\left(x - \lambda\sqrt{\frac{M}{M-1}}\right) dx \left[\phi(x)\right]^{M-1} \tag{4.31}$$

Using (3.24), this can be expressed in terms of Communication Efficiency, β, and M as

$$P_D(\beta;\alpha_R) = \int_{-\infty}^{\infty} G\left(x - \sqrt{\frac{\beta M \log_2 M}{M-1}}\right)\left[\phi(x)\right]^{M-1} dx \tag{4.32}$$

Plots of $P_D(\lambda;\alpha_R)$ vs. λ for various M are given in Figure 5.2 in the next chapter.

Example 2. As a slight generalization of the previous example, we take

$$(S_i \cdot S_j) = \gamma \qquad \text{for all } i \neq j .$$

which is the equi-correlated signal set. Denote its inner product matrix by

$$\alpha_\gamma = \begin{pmatrix} 1 & & \gamma \\ & \ddots & \\ \gamma & & 1 \end{pmatrix} \tag{4.33}$$

Note that $\gamma \geq \frac{-1}{M-1}$, because

$$0 \leq \left(\sum_{i=1}^{M} S_i\right)^* \left(\sum_{j=1}^{M} S_j\right) = M + M(M-1)\,\gamma .$$

Also α_γ characterizes linearly independent signal sets for all γ such that $1 > \gamma > \frac{-1}{M-1}$.

Lemma 4.2: The distance ρ from the origin to the M-1 flat generated by M vector tips which are linearly independent is

$$\rho^2 = \frac{D(\alpha)}{\sum_{i=1}^{M}\sum_{j=1}^{M} C_{ij}(\alpha)} \tag{4.34}$$

where $D(\alpha)$ is the determinant of α and $C_{ij}(\alpha)$ is the cofactor of the ij^{th} element of α.

Proof:

$$\left(S^* e_N\right)^* = e_N^* S = \bar{\rho}^* = (1, 1, \ldots, 1)\ \rho$$

Since S is an M-by-M matrix in this case with $D(S) \neq 0$, we can write

$$e_N^* = \bar{\rho}^* S^{-1}$$

from which

$$e_N = (S^{-1})^* \ \bar{\rho}$$

Then

$$e_N^* e_N = 1 = \bar{\rho}^* S^{-1} S^{-1^*} \ \bar{\rho} = \rho^2 (1, 1, \ldots, 1) S^{-1} S^{-1^*} (1, \ldots, 1)^*$$

But

$$S^{-1} S^{-1^*} = (S^* S)^{-1} = \alpha^{-1}$$

Hence

$$\rho^2 = \frac{1}{(1, \ldots, 1)\ (\alpha)^{-1} (1, \ldots, 1)^*} = \frac{1}{\sum_{i=1}^{M}\sum_{j=1}^{M} (\alpha_{ij})^{-1}} = \frac{D(\alpha)}{\sum_{i=1}^{M}\sum_{j=1}^{M} C_{ij}(\alpha)}$$

QED

In our example:

$$D(\alpha_\gamma) = (1-\gamma)^{M-1} \left[1 + (M-1)\gamma \right] \tag{4.35}$$

$$C_{ii} = \left[1 + (M-2)\gamma \right] (1-\gamma)^{M-2} \tag{4.36}$$

$$C_{ij} = (-\gamma)(1-\gamma)^{M-2} \qquad i \neq j \tag{4.37}$$

from which

$$\rho^2 = \frac{1}{M}\left[1 + (M-1)\gamma\right]$$

Again using the same procedure as in the previous example

$$\left(S'_i \cdot S'_j\right) = \frac{\left(R_i \cdot R_j\right)}{1-\rho^2} = \frac{\gamma-\rho^2}{1-\rho^2} = \frac{-\frac{1}{M}(1-\gamma)}{\left(1-\frac{1}{M}\right)(1-\gamma)} = \frac{-1}{M-1} \qquad \text{for all } i \neq j, \tag{4.38}$$

which again agrees with the regular simplex. Substituting and using $(1-\rho^2) = \left(\frac{M-1}{M}\right)(1-\gamma)$, yields

$$P_D(\lambda;\alpha_\gamma) = P_D\left(\lambda\sqrt{\left(\frac{M-1}{M}\right)(1-\gamma)}\ ;\ \alpha_R\right) \tag{4.39}$$

or equivalently, using Example 1

$$P_D(\lambda;\alpha_\gamma) = P_D\left(\lambda\sqrt{1-\gamma};\ \alpha_o\right) \tag{4.40}$$

4.3 Gradient of the Probability of Detection

We now determine the gradient of $P_D(\lambda;\alpha)$, which is a fundamental quantity in the analysis. The gradient will be used to determine necessary conditions which signal sets must satisfy to be a local optimum. A general result obtained in this section will further reduce the size of the class of signal sets which contain the optimal sets.

From (4.13)

$$P_D(\lambda;\alpha) = \frac{1}{M} e^{-\frac{1}{2}\lambda^2} E\left\{e^{\lambda \max_i \xi_i}\right\}$$

Let

$$x = \max_i \xi_i$$

and

$p_M(x;\alpha)$ be the probability density function of x.

Then

$$P_D(\lambda;\alpha) = \frac{1}{M} e^{-\frac{1}{2}\lambda^2} \int_{-\infty}^{\infty} e^{\lambda x} p_M(x;\alpha)\, dx \tag{4.41}$$

Define

$$\Phi(x;\alpha) = \int_{-\infty}^{x} p_M(y;\alpha)\, dy = \Pr\left[\max_i \xi_i = y \le x\right] = \underbrace{\int_{-\infty}^{x} \cdots \int}_{M\text{-fold}} G(\xi;o;\alpha)\, d|\xi| \tag{4.42}$$

Substituting

$$\phi(\lambda;\alpha) = M P_D(\lambda;\alpha) e^{\lambda^2/2} = \int_{-\infty}^{\infty} e^{\lambda x} \frac{d}{dx} \Phi\left[(x;\alpha)\right] dx$$

$$= \int_{-\infty}^{\infty} e^{\lambda x} \frac{d}{dx}\left[\Phi(x;\alpha) - \{\phi(x)\}^M\right] dx + M\int_{-\infty}^{\infty} e^{\lambda x} \left[\phi(x)\right]^{M-1} G(x)\, dx \tag{4.43}$$

The second integral is not a function of α. Integrating the first integral by parts gives

$$\phi(\lambda;\alpha) = e^{\lambda x}\left[\Phi(x;\alpha) - \{\phi(x)\}^M\right]\Bigg|_{-\infty}^{\infty} - \lambda\int_{-\infty}^{\infty} e^{\lambda x}\left[\Phi(x;\alpha) - \{\phi(x)\}^M\right] dx$$

$$+ M\int_{-\infty}^{\infty} G(x)\left[\phi(x)\right]^{M-i} dx \tag{4.44}$$

As $x \to \pm\infty$, $\Phi(x;\alpha) - \left[\phi(x)\right]^M \to o$ as $e^{-\frac{1}{2}x^2}$. Thus the first term in (4.44) vanishes.

Now take the derivative of $\phi(\lambda;\alpha)$ with respect to λ_{12}.

Thus

$$\frac{\partial \phi(\lambda;\alpha)}{\partial \lambda_{12}} = -\lambda \int_{-\infty}^{\infty} e^{\lambda x} \frac{\partial \Phi(x;\alpha)}{\partial \lambda_{12}} dx$$

$$= -\lambda \int_{-\infty}^{\infty} e^{\lambda x} \frac{\partial}{\partial \lambda_{12}} \left[\int \overset{x}{\cdots} \int_{-\infty} G(\xi;0;\alpha)\, d|\xi| \right] dx$$

$$= -\lambda \int_{-\infty}^{\infty} e^{\lambda x} \left[\int \overset{x}{\cdots} \int_{-\infty} \left\{ \frac{\partial}{\partial \lambda_{12}} G(\xi;o;\alpha) \right\} d|\xi| \right] dx \tag{4.45}$$

Denote the Characteristic Function of $G(\xi;o;\alpha)$ by $C(t;\alpha)$, where t is a M-by-1 column vector. Then

$$C(t;\alpha) = e^{-\frac{1}{2} t^* \alpha t} \tag{4.46}$$

and

$$G(\xi;o;\alpha) = \mathfrak{F}^{-1}\left[C(t;\alpha)\right]$$

where $\mathfrak{F}^{-1}\{\ \}$ indicates the M-dimensional inverse Fourier Transform.

Thus

$$G(\xi;o;\alpha) = \int \overset{\infty}{\cdots} \int_{-\infty} e^{it^*\xi} \frac{e^{-\frac{1}{2} t^* \alpha t}}{(\sqrt{2\pi})^M} d|t| \tag{4.47}$$

Using this

$$\frac{\partial G(\xi;o;\alpha)}{\partial \lambda_{12}} = -\int \overset{\infty}{\cdots} \int_{-\infty} t_1 t_2\, e^{it^*\xi} \frac{e^{-\frac{1}{2} t^* \alpha t}}{(\sqrt{2\pi})^M} d|t|$$

$$= \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \int \overset{\infty}{\cdots} \int_{-\infty} e^{it^*\xi} \frac{e^{-\frac{1}{2} t^* \alpha t}}{(\sqrt{2\pi})^M} d|t|$$

$$= \frac{\partial^2}{\partial \xi_1 \partial \xi_2} G(\xi;o;\alpha)$$

or, using slightly different notation

$$\frac{\partial G(\xi;o;\alpha)}{\partial\lambda_{12}} = \frac{\partial^2 G(\xi_1,\ldots,\xi_M;o;\alpha)}{\partial\xi_1\,\partial\xi_2} \tag{4.48}$$

Substituting:

$$\frac{\partial\phi(\lambda;\alpha)}{\partial\lambda_{12}} = -\lambda\int_{-\infty}^{\infty} e^{\lambda x}\,dx\left[\underbrace{\int_{-\infty}^{x}\!\!\cdots\!\int}_{\text{M-fold}} \frac{\partial^2 G(\xi_1,\ldots,\xi_M;o;\alpha)}{\partial\xi_1\,\partial\xi_2}\,d\,|\xi|\right]$$

$$= -\lambda\int_{-\infty}^{\infty} e^{\lambda x}\,dx\left[\underbrace{\int_{-\infty}^{x}\!\!\cdots\!\int}_{\text{M-2 fold}} G(x,x,\xi_3,\ldots,\xi_M;o;\alpha)\,d\xi_3\ldots d\xi_M\right] \tag{4.49}$$

The integral is ≥ 0 for all λ. Therefore

$$\frac{\partial\phi(\lambda;\alpha)}{\partial\lambda_{12}} \leq 0 \text{ for all } \lambda > o. \tag{4.50}$$

Similarly

$$\frac{\partial\phi(\lambda;\alpha)}{\partial\lambda_{ij}} \leq 0 \text{ for all } \lambda > o, \text{ and all } i \neq j. \tag{4.51}$$

Hence, decreasing λ_{ij} increases $\phi(\lambda;\alpha)$, and we have proven the following.

Theorem 4.3:

If $\lambda'_{ij} \leq \lambda_{ij}$ for all $i \neq j$ (4.52)

then

$$P_D(\lambda;\alpha') \geq P_D(\lambda;\alpha) \text{ for all } \lambda > 0. \tag{4.53}$$

This proves that when determining the optimum α, we should make the $\{\lambda_{ij}\}$ as small as possible, which corresponds to placing the set of signal vectors as far apart from one another as possible within the restrictions imposed by the covariance matrix. This is what one intuitively would expect. This gives a partial ordering of the class of admissible signal sets, which clearly is not a complete ordering. Hence, this result reduces the class of sets which contain the optimal sets, but does not tell precisely which sets are optimum. There is also no dimensionality restriction in this result.

In particular, if

$$\alpha_{\rho_o} = \max_{i \neq i} \{\lambda_{ij}\}$$

then, denoting the corresponding inner product matrix by $\alpha\rho_o$, i.e.,

$$\alpha_{\rho_o} = \begin{pmatrix} 1 & & \rho_o \\ & \ddots & \\ \rho_o & & 1 \end{pmatrix}$$

we have

$P_D(\lambda;\alpha) > P_D(\lambda;\alpha\rho_o)$ for all λ, and hence, minimizing ρ_o, which corresponds to maximizing the minimum distance within the admissible class, provides a lower bound in this class and is independent of λ.

4.4 Signal Sets Whose Convex Hull Does Not Include the Origin

Definition: The Convex Hull of a set of M vectors $\{S_i\}$ is the set

$$\pi = \left\{ Y \,\middle|\, Y = \sum_{i=1}^{M} \gamma_i S_i;\ \gamma_i \geq 0;\ \sum_{i=1}^{M} \gamma_i = 1 \right\} \tag{4.54}$$

We must first prove the following theorem, which will be needed in what follows.

Theorem 4.4.

Let $x = \max_i \xi_i$, where $\xi_i = (Y \cdot S_i)$, $i=1, \ldots, M$, and

$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_M \end{pmatrix}$ is a Gaussian variate with zero mean and covariance matrix equal to α. Let $p_M(x;\alpha)$ be the probability density function of x.

Then

$$p_M(x;\alpha) = 0 \text{ for } x < 0 \tag{4.55}$$

if and only if π contains the origin.

Proof:

$0 \epsilon \pi$ implies there exists a set of $\{a_i\}$ such that

$$\sum_{i=1}^{M} a_i S_i = 0, \quad a_i \geq 0, \quad \sum_{i=1}^{M} a_i = 1.$$

First suppose $p_M(x;\alpha) = 0$ for all $x < 0$.

Consider the

$$\text{minimum}\left\{ \left\| \sum_{i=1}^{M} a_i S_i \right\| \; \middle| \; \sum_{i=1}^{M} a_i = 1, \; a_i \geq 0 \right\}$$

over all possible sets of $\{a_i\}$. That is, consider the point closest to the origin of the closed convex hull generated by the $\{S_i\}$. Let

$$\delta = \min \left\| \sum_{i=1}^{M} a_i S_i \right\|$$

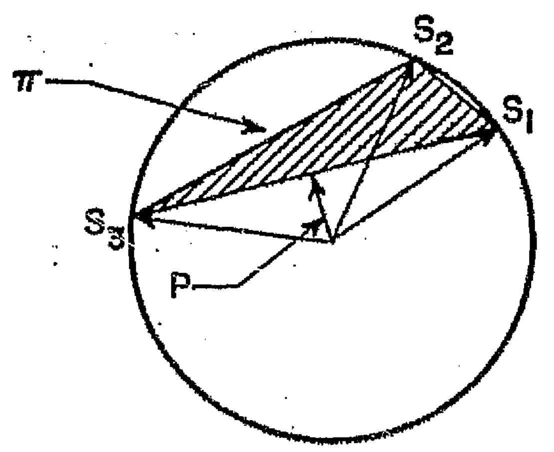
Suppose $\delta > 0$. We have then satisfied all the hypotheses of the following Lemma from Convex Body Theory (illustrated in Figure 4.4).

Lemma 4.3: Let S equal a closed convex set in E_D, with $0 \notin S$. Then there exists a vector $p \epsilon E_D$ such that

$$(p \cdot x) > 0 \text{ for all } x \epsilon S.$$

Applying this, call the vector which satisfies the Lemma V_o and adjust its magnitude so that

$$\sqrt{V_o^* V_o} = ||V_o|| = \delta$$



Example of a Signal Structure Whose Convex Hull
Does Not Include the Origin

FIGURE 4. 4

Then by the Lemma

$$(V_o \cdot S_i) > 0, \; i = 1, \ldots, M.$$

Now take

$$\Delta = \left\{ Y \;\middle|\; Y \epsilon E_D;\; x = \max_i \, (Y \cdot S_i) < 0 \right\}$$

Δ is not empty because $-V_o \; \epsilon \Delta$. Because of the strict inequality in the above Lemma, $(-V_o)$ is in the interior of Δ and it thus has non-zero probability, which is a contradiction. Therefore, δ must be zero.

Conversely, assume $\delta = 0$, which implies that there exists $a_i \geq 0$ with $\sum_{i=1}^{M} a_i = 1$, such that

$$\sum_{i=1}^{M} a_i S_i = 0$$

This implies that for every $Y \epsilon E_D$ not all of the $(Y \cdot S_i)$ can have the same sign, because

$$\sum_{i=1}^{M} a_i \, (Y \cdot S_i) = Y \cdot \left(\sum_{i=1}^{M} a_i S_i \right) = 0.$$

Hence at least one of the $(Y \cdot S_i)$ must be positive, that is

$$x = \max_i (Y \cdot S_i) \geq 0 \quad \text{for all } Y.$$

Therefore

$$p_M (x;\alpha) = 0 \text{ for all } x < 0.$$

QED

Note that this proof is true for any M and any D. We now prove the main theorem of this section.

Theorem 4.5.

If the convex hull generated by a set of $\{S_i, i=1,\ldots,M\}$ does not include the origin, there exists a signal set whose convex hull does include the origin with a higher probability of detection for all signal-to-noise ratios.

Proof:

Let α correspond to a set of M linearly dependent vectors $\{S_i\}$. From Theorem 4.2, we know that all optimal signal sets are contained in the class which are linearly dependent, and we can thus restrict our attention to this class. Assume the convex hull generated by the $\{S_i\}$ does not contain the origin.

Assume the M vectors span E_{M-1}. Then, since the convex hull generated by the $\{S_i\}$ does not contain the origin, from the Lemma there exists an M-2 flat through the tips of M-1 of the $\{S_i\}$, say $S_1, \ldots, S_{M-1}$, which separates the origin from the remaining vector S_M, as illustrated for 3 signals in 2-space in Figure 4.5. Let the flat have equation

$$(Y \cdot e_N) = \rho$$

where $\rho > 0$ is the distance from the origin to the flat, and e_N is the unit normal to the flat. Then

$$(S_i \cdot e_N) = \rho, \quad i = 1, \ldots, M-1,$$

and

$$(S_M \cdot e_N) > \rho.$$

Now define a new set of fectors $\{S_i'\}$ such that

$$S_M' = S_M - 2e_N(e_N \cdot S_M)$$

and

$$S_i' = S_i, \quad i = 1, \ldots, M-1. \tag{4.56}$$

The S_i' are generated by choosing the S_i' the same as the S_i for those S_i whose tips generate the M-2 flat, and forming S_M' from S_M by choosing the component of S_M perpendicular to the flat, and taking its negative, and leaving the components of S_M in the flat unchanged, as illustrated in Figure 4.6.

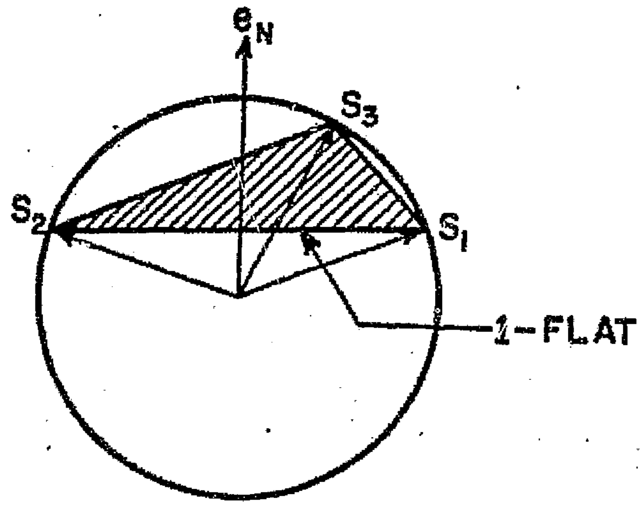


FIGURE 4.5

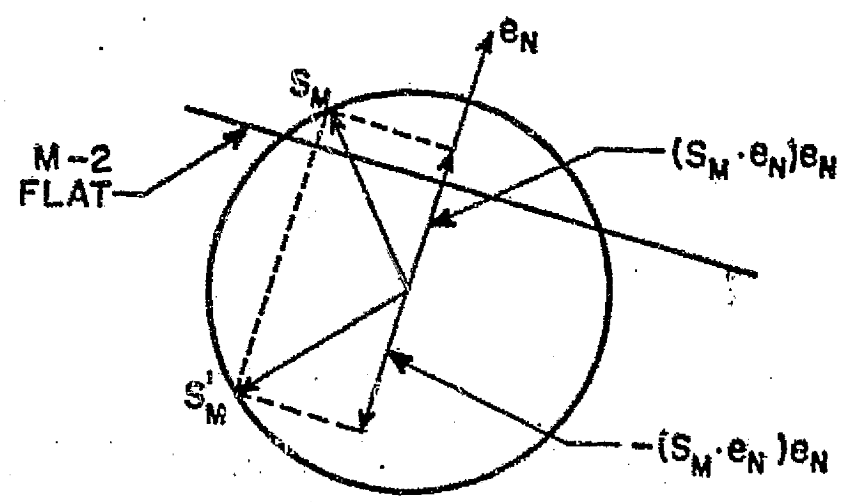


FIGURE 4.6

We have

$$(S'_i \cdot S'_i) = 1, \quad i = 1, \ldots, M$$

Also

$$(S'_i \cdot S'_M) = \left(S_i \cdot \left[S_M - 2\, e_N (e_N \cdot S_M) \right] \right)$$

$$= (S_i \cdot S_M) - 2(S_i \cdot e_N)\,(e_N \cdot S_M)$$

Since

$$(S_i \cdot e_N) = \rho > 0$$

and

$$(e_N \cdot S_M) > \rho > 0$$

then

$$\lambda'_{iM} = (S'_i \cdot S'_M) < \lambda_{iM} - 2\,\rho^2 < \lambda_{iM}, \quad i = 1, \ldots, M-1.$$

Since

$$\lambda'_{ij} = \lambda_{ij} \text{ for all } i \neq j;\ i,\ j < M,$$

we can conclude, using Theorem 4.3, that

$$P_D\,(\lambda;\, \alpha') \geq P_D\,(\lambda;\alpha) \text{ for all } \lambda > 0.$$

If the dimensionality of the $\{S_i\}$ is $D < M-1$, and the convex hull generated by the $\{S_i\}$ does not include the origin, there exists a D-1 flat through the tips of D of the $\{S_i\}$, say $(S_1, \ldots, S_D)$ such that

$$(e_N \cdot S_i) = \rho \qquad i = 1, \ldots, D$$

and

$$(e_N \cdot S_i) > \rho \qquad i = D+1, \ldots, M$$

where again ρ is the distance from the origin to the flat and e_N is the unit normal to the flat.

Define $\{S'_i\}$ such that

$$S'_i = S_i, \quad i = 1, \ldots, D$$

and

$$S'_i = S_i - 2e_N(e_N \cdot S_i) \quad i = D+1, \ldots, M.$$

Again we can conclude

$$P_D(\lambda;\alpha') \geq P_D(\lambda;\alpha) \text{ for all } \lambda > 0.$$

If the convex hull generated by the $\{S'_i\}$ does not contain the origin, repeat the procedure. Successive repetitions of the procedure will result in a signal set whose convex hull contains the origin. A maximum of D-1 iterations will be needed for the hull of the resultant signal set to contain the origin. Since the above inequality is true for each iteration, the proof is complete.

QED

We insert at this point the following Lemma concerning the behavior of the components of the gradient vector, which will be used extensively in later chapters. It contains an important property which the gradient of the optimal set must possess.

Define

$$\phi_{ij}(\lambda;\alpha) = -\frac{M}{\lambda}\frac{\partial \phi(\lambda;\alpha)}{\partial \lambda_{ij}}. \tag{4.57}$$

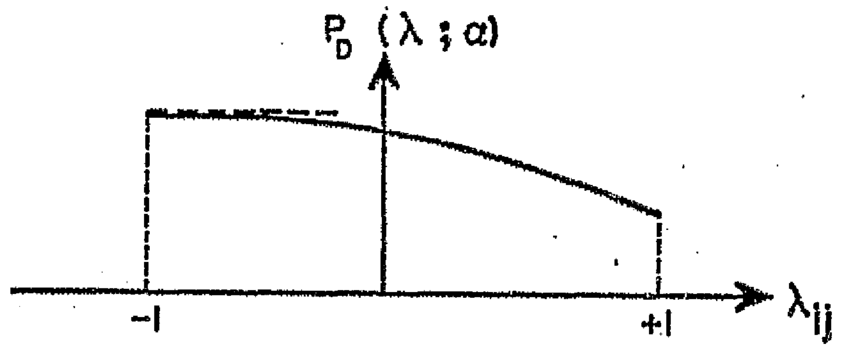
Lemma 4.4: For those α whose corresponding convex hull contains the origin

$$\phi_{ij}(\lambda;\alpha) = 0 \tag{4.58}$$

if and only if

$$\lambda_{ij} = -1 \tag{4.59}$$

Remark: With the fact that $\frac{\partial \phi(\lambda;\alpha)}{\partial \lambda_{ij}} \leq 0$ for any λ_{ij} and this Lemma, we can conclude that as a function of λ_{ij}, the behavior of $P_D(\lambda;\alpha)$ is as shown in Figure 4. 7, where the derivative with respect to λ_{ij} is zero only at $\lambda_{ij} = -1$.



$P_D\ (\lambda;\alpha)$ vs. λ_{ij}

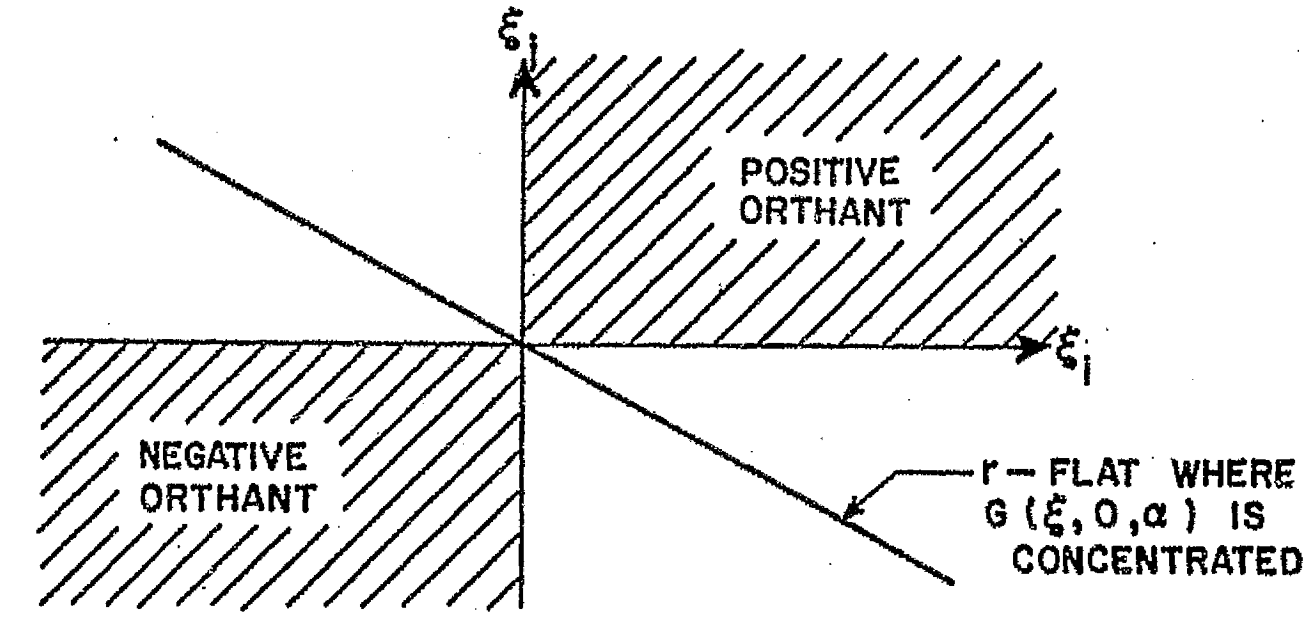
FIGURE 4. 7

Proof: It is sufficient to show

$\phi_{12}\ (\lambda;\alpha) = 0$ if and only if $\lambda_{12} = -1$. From (4. 57) and (4. 49)

$$\frac{\phi_{12}(\lambda;\alpha)}{M} = \int_{-\infty}^{\infty} e^{\lambda x}\, dx \underbrace{\int_{-\infty}^{x}\cdots\int}_{M-2\text{ fold}} G\ (x, x, \xi_3, \ldots, \xi_M; 0;\alpha)\, d\xi_3 \ldots d\xi_M \tag{4. 60}$$

From Theorem 4. 2, we know that the Gaussian density function in (4. 60) is concentrated in an r-flat where $r \leq M-1$. Also from Theorem 4. 4, $p_M\ (x;\alpha) = 0$ for $x < 0$, from which we can conclude that the r-flat cannot be located in the negative orthant, where the ξ_i are all less than zero. From which it is immediate that the r-flat also cannot be in the positive orthant where the ξ_i are all greater than zero. Thus, the r-flat must be located as in Figure 4. 8, where it necessarily intersects the origin since the means of all the ξ_i are zero.



Position of r-Flat Containing $G(\xi;0;\alpha)$

FIGURE 4. 8

From Theorems 4. 4 and 4. 5, for optimal sets the integral over negative x in (4. 60) vanishes, and can be written

$$\frac{\phi_{12}(\lambda;\alpha)}{M} = \int_0^\infty e^{\lambda x} dx \underbrace{\int_{-\infty}^{x} \cdots \int}_{M-2 \text{ fold}} G(x,\ x,\ \xi_3, \ldots, \xi_M; 0;\alpha)\, d\xi_3 \ldots d\xi_M \tag{4.61}$$

Assume first that $\lambda_{12} = -1$. This implies that $\xi_1 = -\xi_2$ with probability 1. But in the integration over x in (4.61) we are integrating only over points where $\xi_2 = \xi_1 = x$. Hence (see Figure 4. 8), at every $x > 0$ the Gaussian density is identically zero, and we can conclude

$$\phi_{12}(\lambda;\alpha) = 0.$$

Conversely, assume $\phi_{12}(\lambda;\alpha) = 0$.

Further assume $\lambda_{12} > -1$, from which we want to arrive at a contradiction. To do this, write the Gaussian density in (4. 61) as a conditional density in the following way

$$G(\xi_1 = x,\ \xi_2 = x,\ \xi_3, \ldots, \xi_M; 0; \alpha) =$$
$$G\left(\xi_3, \ldots, \xi_M / \xi_1 = x,\ \xi_2 = x\right)\ G(\xi_1 = x,\ \xi_2 = x) \tag{4.62}$$

where we call

$$G(\xi_1 = x,\ \xi_2 = x) = G_{12}(x) = \frac{e^{-\frac{x^2}{1+\lambda_{12}}}}{2\pi\sqrt{1-\lambda_{12}^2}} \tag{4.63}$$

and the conditional Gaussian density has mean

$$E\left(\xi_3, \ldots, \xi_M \middle/ \xi_1 = x,\ \xi_2 = x\right) = \left(\frac{\lambda_{13}+\lambda_{23}}{1+\lambda_{12}}\, x,\ \ldots,\ \frac{\lambda_{1M}+\lambda_{2M}}{1+\lambda_{12}}\, x\right) \tag{4.64}$$

and covariance matrix equal to

$$\begin{pmatrix} 1 & \lambda_{34} & \cdots & \lambda_{3M} \\ \lambda_{34} & \ddots & & \vdots \\ \vdots & & \ddots & \lambda_{M-1,M} \\ \lambda_{3M} & \cdots & \lambda_{M-1,M} & 1 \end{pmatrix} - \begin{pmatrix} \lambda_{13} & \lambda_{23} \\ \vdots & \vdots \\ \lambda_{1M} & \lambda_{2M} \end{pmatrix} \frac{\begin{pmatrix} 1 & -\lambda_{12} \\ -\lambda_{12} & 1 \end{pmatrix}}{1-\lambda_{12}^2} \begin{pmatrix} \lambda_{13} & \cdots & \lambda_{1M} \\ \lambda_{23} & \cdots & \lambda_{2M} \end{pmatrix} \tag{4.65}$$

(For reference, a summary of conditional Gaussian densities is given in Appendix A).

Substituting:

$$\frac{\phi_{12}(\lambda;\alpha)}{M} = \int_c^{\infty} e^{\lambda x}\, G_{12}(x)\, F_{12}(x)\, dx \tag{4.66}$$

where

$$F_{12}(x) = \int_{-\infty}^{x} \cdots \int G\left(\xi_3, \ldots, \xi_M \middle/ \xi_1 = x,\ \xi_2 = x\right) d\xi_3 \ldots d\xi_M \tag{4.67}$$

When $\lambda_{12} > -1$, $G_{12}(x)$ has positive measure for all x. Also $F_{12}(x) \geq 0$ for all x. Therefore, if we can show that there exists some x for which $F_{12}(x)$ has positive measure, then we could conclude that

$$\phi_{12}(\lambda;\alpha) > 0$$

contradicting the original assumption.

To do this, let

$$\hat{\xi}_j = \xi_j - E\left[\xi_j / \xi_1 = x,\ \xi_2 = x\right] = \xi_j - \left(\frac{\lambda_{1j} + \lambda_{2j}}{1 + \lambda_{12}}\right) x, \quad j=3, \ldots, M. \tag{4.68}$$

Then

$$F_{12}(x) = \int_{-\infty}^{\alpha_3 x} \cdots \int^{\alpha_M x} \hat{G}\left(\hat{\xi}_3, \ldots, \hat{\xi}_M\right) d\hat{\xi}_3 \ldots d\hat{\xi}_M \tag{4.69}$$

where the $\hat{\xi}_j$'s have zero mean and covariance matrix given by (4.65), and $\alpha_j = 1 - \frac{\lambda_{1j} + \lambda_{2j}}{1 + \lambda_{12}}$, $j = 3, \ldots, M$.

If the p. d. f. $\hat{G}(\hat{\xi}_3, \ldots, \hat{\xi}_M)$ is nonsingular, then $F_{12}(x)$ has positive measure for all x, and the proof is complete. We know that this is not true, however, because $F_{12}(x)$ would then have positive measure for negative x, hence implying $p_M(x;\alpha) > 0$ for $x < 0$, which is a contradiction. Therefore, the density is singular, and, as before, it is concentrated in an r-flat through the origin, where $r < M-2$. Similarly the r-flat cannot be in the negative or positive orthant. Thus it is again in a position as indicated in Figure 4.8 for the $\{\xi_j\}$.

In this case, to show that $F_{12}(x)$ has positive measure for some x, we must show that the direction taken by the line segment $(\alpha_3 x, \ldots, \alpha_M x)$, as x varies over $(0, \infty)$ is such that the negative orthant consisting of all $\hat{\xi}_i < \alpha_i x$, $i=3, \ldots, M$, intersects the r-flat for some x. We claim that this is always the case, based on the following reasoning:

Suppose there is no x, as x varies over $(0, \infty)$, such that the negative orthant below $(\alpha_3 x, \ldots, \alpha_M x)$ intersects the r-flat containing the singular Gaussian distribution. This situation is illustrated in Figure 4.9.

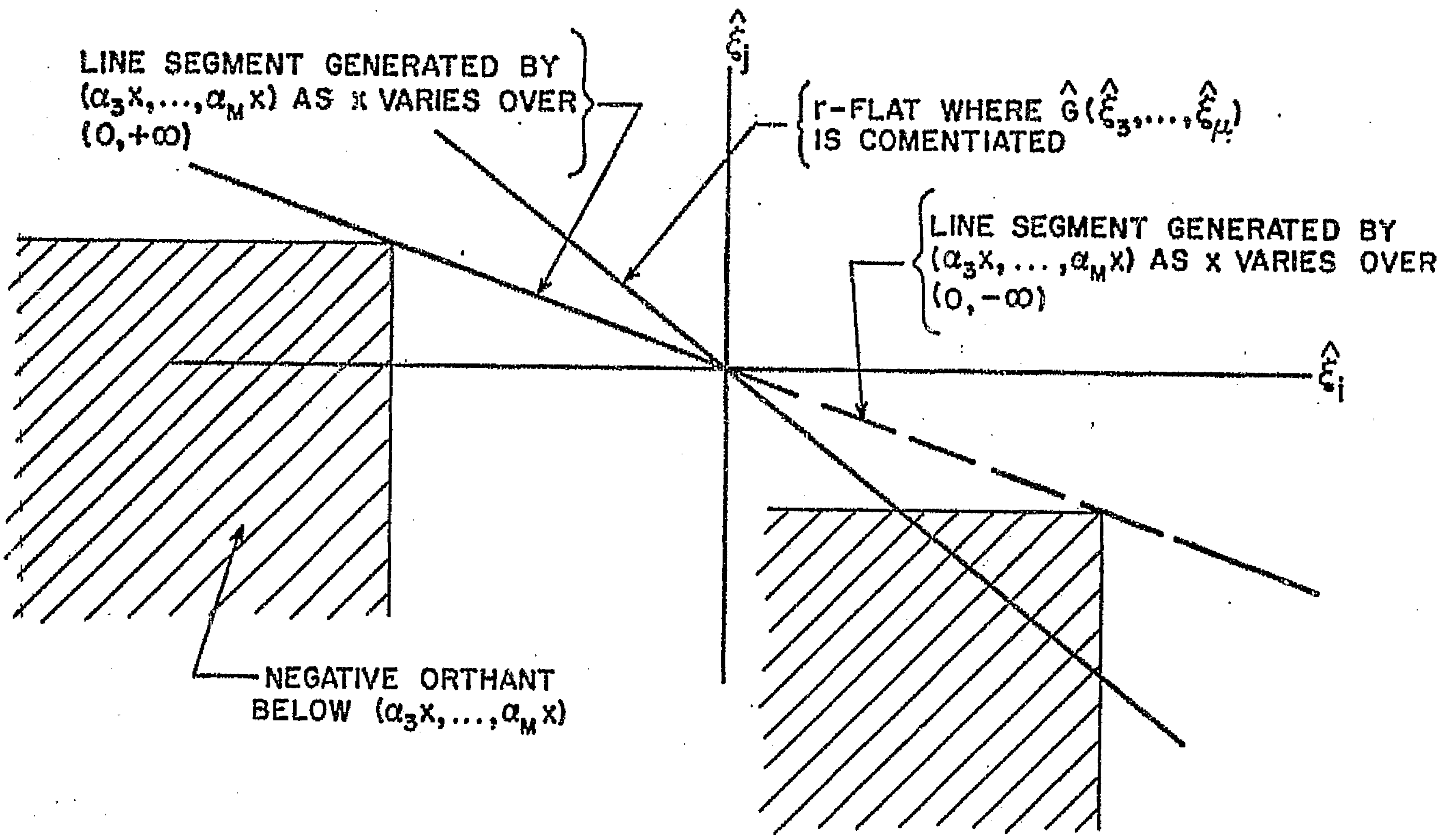


FIGURE 4. 9

In this case, if we then consider the integral over negative x (the dotted line in Figure 4. 9) it is immediate that the negative orthant below points on this line intersects the r-flat, which implies $p_M(x;\alpha) > 0$ for $x < 0$, contradicting the assumption that the convex hull of the corresponding signal set contains the origin.

Therefore, we can conclude that $F_{12}(x)$ necessarily has positive measure for $x > 0$ and $\phi_{12}(\lambda;\alpha) > 0$ contradicting the original assumption. Thus $\phi_{12}(\lambda;\alpha) = 0$ implies $\lambda_{12} = -1$.

QED

4.5 The Admissible α-Space

It has already been shown that the admissible α-space, Γ, consists of those M-by-M symmetric non-negative definite matricies with 1's along the main diagonal and all off-diagonal elements such that $|\lambda_{ij}| \leq 1$.

We now mention some of the pertinent properties of the α-space.

First note that Γ is bounded, for if we define the magnitude of a matrix in the usual way, by $|\mathrm{tr}\,[\alpha^* \alpha]|$, where tr means "the trace of", then

$$||\alpha||^2 = |\mathrm{tr}\,[\alpha^* \alpha]| \leq M^2. \tag{4.70}$$

Γ is also convex, for if α_1 and α_2 are in Γ, then

$$\alpha^o = \beta \alpha_1 + (1-\beta)\alpha_2 \qquad 0 \leq \beta \leq 1 \tag{4.71}$$

is of the form

$$\begin{pmatrix} 1 & \ddots & \lambda^o_{ij} \\ & \ddots & \\ \lambda^o_{ji} & \ddots & 1 \end{pmatrix}$$

and is non-negative definite, since for any M-by-1 vector t

$$t^* \alpha^o t = \beta t^* \alpha_1 t + (1-\beta)\, t^* \alpha_2\, t \geq 0 \tag{4.72}$$

Hence

$$\alpha^o \in \Gamma.$$

It can similarly be shown that the class of positive definite α is also convex. Therefore, the interior of Γ consists of α which are positive definite, for which $D(\alpha) > 0$ and the boundary consists of those admissible α for which $D(\alpha) = 0$. Therefore we know that all the optimal α lie on the boundary of Γ.

It can also be shown that Γ is closed. Further, its surface is smooth since $D(\alpha) = 0$ is a polynomial in the set of $\{\lambda_{ij}\}$.

<u>Lemma 4.5:</u>

1) $$\sum_{i=1}^{M} S_i = 0 \text{ if and only if } \sum_{i=1}^{M} \lambda_{ij} = 0, \; j=1, \ldots, M. \tag{4.73}$$

2) $\sum_{i=1}^{M} \lambda_{ij} = 0, \ j=1, \ldots, M,$ if and only if $\sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_{ij} = 0.$ (4.74)

Proof:

Assume $\sum_{i=1}^{M} S_i = 0.$ Then $\sum_{i=1}^{M} S_i \cdot S_j = \sum_{i=1}^{M} \lambda_{ij} = 0, j=1, \ldots, M,$

and hence

$$\sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_{ij} = 0.$$

Conversely

$$\sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_{ij} = \sum_{i=1}^{M} \sum_{j=1}^{M} S_i \cdot S_j = \left(\sum_{i=1}^{M} S_i\right) \cdot \left(\sum_{j=1}^{M} S_j\right)$$

If $\sum_{i=1}^{M} S_i = A$ where A is not the zero vector, then $A^{*}A > 0$ and

$\sum_{j=1}^{M} \sum_{i=1}^{M} \lambda_{ij} > 0,$ contradicting the original assumption.

QED

Therefore the class of signals for which

$$\sum_{i=1}^{M} S_i = 0 \tag{4.75}$$

is identical to the class for which

$$\sum_{j=1}^{M} \sum_{i=1}^{M} \lambda_{ij} = 0.$$

Note also that this class is a convex set, and that α for which

$$\sum_{j=1}^{M} \sum_{i=1}^{M} \lambda_{ij} = 0$$

is a hyperplane in the $\frac{M(M-1)}{2}$ Euclidean space containing Γ.

If $\sum \lambda_{ij} = 0$, $j=1, \ldots, M$, then the column vector $(1, \ldots, 1)^*$ is an eigenvector with corresponding eigenvalue equal to zero, because

$$\left(\alpha\right)\begin{pmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{pmatrix} = 0\begin{pmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{pmatrix} \tag{4.76}$$

Since $D(\alpha)$ is the product of the eigenvalues of α, we know that all admissible signal sets for which

$$\sum_{i=1}^{M}\sum_{j=1}^{M} \lambda_{ij} = 0 \tag{4.77}$$

all lie on the surface of Γ, because they have an eigenvalue equal to zero. Note that this hyperplane defined by (4.76) contains non-admissible as well as admissible α. The regular simplex signal structure, α_R, and the optimal signal set when D=2, namely equal spacing, both lie in this hyperplane. Figure 4.10 is an illustration of the α-space.

The following relationship will be used extensively in the next two chapters.

<u>Lemma 4.6:</u>

For any symmetric matrix

$$\frac{\partial D(\alpha)}{\partial \lambda_{ij}} = 2\, C_{ij} \tag{4.78}$$

where C_{ij} is the cofactor including sign of λ_{ij}.

<u>Proof:</u>

Expanding $D(\alpha)$ in terms of the j^{th} column yields

$$D(\alpha) = \sum_{k=1}^{M} \lambda_{kj}\, C_{kj}.$$

Then

$$\frac{\partial D(\alpha)}{\partial \lambda_{ij}} = C_{ij} + \sum_{\substack{k=1\\ k\neq j}}^{M}{}' \lambda_{kj}\, \frac{\partial C_{kj}}{\partial \lambda_{ij}}$$

Now

$$\left.\frac{\partial C_{kj}}{\partial \lambda_{ij}}\right|_{k\neq j} = \frac{\partial}{\partial \lambda_{ij}} \left[\sum_{\substack{l=1 \\ l\neq k}}^{M} \lambda_{li} \; C_{li}^{kj} \right] = C_{ji}^{kj}$$

where C_{li}^{kj} is the cofactor of the cofactor, i. e., the determinant resulting after the k^{th} and l^{th} row and the i^{th} and j^{th} column have been removed from α. Substituting

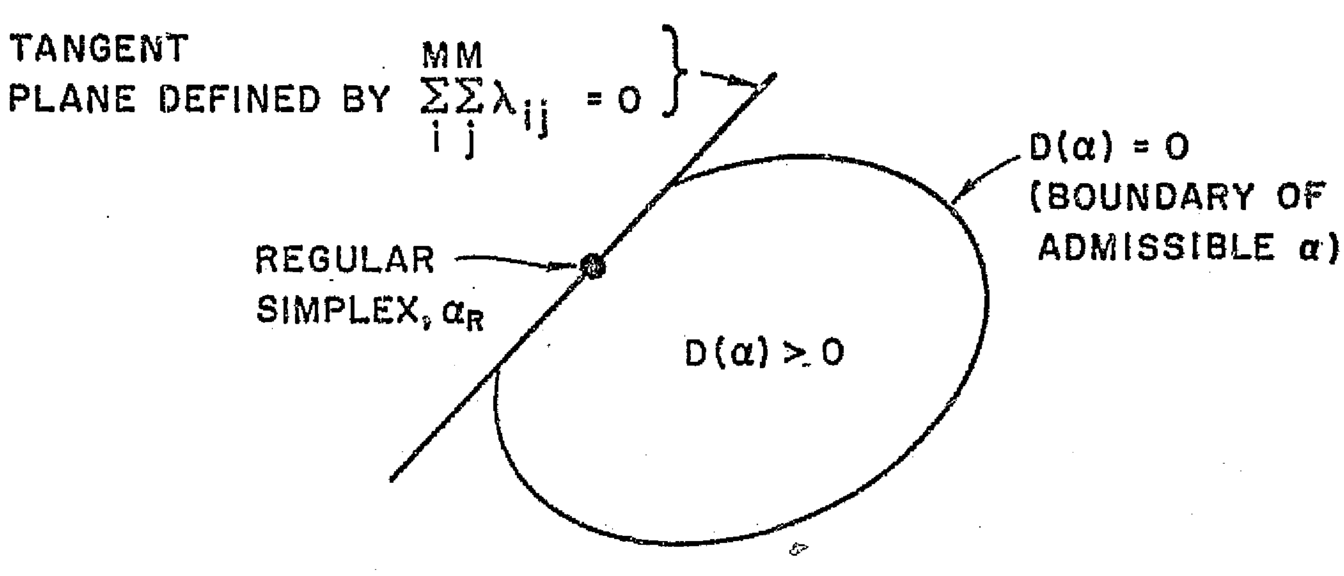
$$\frac{\partial D(\alpha)}{\partial \lambda_{ij}} = C_{ij} + \sum_{\substack{k=1 \\ k\neq i}}^{M} \lambda_{kj} \; C_{ji}^{kj}$$

Using the fact that $C_{ji}^{kj} = C_{jk}^{ij}$, which results from α being symmetric, the summation becomes C_{ij}.

<u>QED</u>

With this Lemma and noting that for the regular simplex the C_{ij} are independent of i and j, it can be concluded that the tangent plane to the surface defined by $D(\alpha) = 0$ at α_R is given by

$$\sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_{ij} = 0$$



THE $\frac{M(M-1)}{2}$ DIMENSIONAL α-SPACE

FIGURE 4.10

4.6 Series Expansions and Asymptotic Approximations

In this section we use Hermite polynomial expansions for the probability of detection for large M and large λ. A summary of these expansions, known as tetra-choric series, is given in Appendix B. For further details on tetra-choric series, see Reference 4.7. The asymptotic expansion that is of most interest to us can be stated in the following:

Theorem 4.6: For large λ and fixed M, the probability of detection can be asymptotically approximated by

$$P_D(\lambda;\alpha) = 1 - \frac{1}{M}\sum_{i}^{M}\sum_{\neq j}^{M} \frac{e^{-\frac{1}{2}\gamma_{ij}^2\lambda^2}}{\sqrt{2\pi}\,\gamma_{ij}\,\lambda} \tag{4.79}$$

where

$$\gamma_{ij} = \sqrt{\frac{1-\lambda_{ij}}{1+\lambda_{ij}}} \tag{4.80}$$

Proof: From (4.41), we have

$$P_D(\lambda;\alpha) = \frac{1}{M} e^{-\frac{1}{2}\lambda^2} \int_{-\infty}^{\infty} e^{\lambda x}\, p_M(x;\alpha)\, dx$$

For $p_M(x;\alpha)$ we can write

$$\begin{aligned} p_M(x;\alpha) &= \sum_{i=1}^{M} p\left(\xi_i = x\right)\ p\left[\xi_j < \xi_i \text{ for all } j\neq i \middle/ \xi_i = x\right] \\ &= \sum_{i=1}^{M} G(x)\ p\left[\xi_j < x \text{ for all } j\neq i \middle/ \xi_i = x\right] \end{aligned} \tag{4.81}$$

Now

$$p\left[\xi_j < x \text{ for all } j\neq i \middle/ \xi_i = x\right]$$

$$= \underbrace{\int_{-\infty}^{x}\!\!\cdots\!\int}_{M-1\text{ fold}} p_{M-1}\left(\xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_M \middle/ \xi_i = x\right) d\xi_1\ldots d\xi_{i-1}d\xi_{i+1}\ldots d\xi_M \tag{4.82}$$

Let

$$\bar{\xi}_i = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{i-1} \\ \xi_{i+1} \\ \vdots \\ \xi_M \end{pmatrix} ; \; m_i = E\left(\bar{\xi}_i \middle/ \xi_i = x\right)$$

and

$$R_i = E\left[\bar{\xi}_i \bar{\xi}_i^* \middle/ \xi_i = x \right]$$

Then

$$p_{M-1} \left(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_M \middle/ \xi_i = x\right) = G\,(\bar{\xi}_i;\, m_i;\, R_i) \tag{4.83}$$

and

$$p\left(\xi_j < x \text{ for all } j \neq i \middle/ \xi_i = x\right) = \underbrace{\int_{-\infty}^{x} \cdots \int}_{M-1 \text{ fold}} G(\bar{\xi}_i;\, m_i;\, R_i)\, d\left|\xi_i\right| \tag{4.84}$$

Now let

$$\rho_i = \begin{pmatrix} \lambda_{1i} \\ \vdots \\ \lambda_{i-1,i} \\ \lambda_{i+1,i} \\ \vdots \\ \lambda_{Mi} \end{pmatrix}$$

ρ_i is the i^{th} column of α with $\lambda_{ii} = 1$ removed. Thus ρ_i is a (M-1)-by-1 column vector. Also define α_i as α with the i^{th} row and i^{th} column removed. α_i is the unconditional covarianc^ matrix of $\bar{\xi}_i$. Then from the conditional Gaussian distribution

$$m_i = (\rho_i)\, x \tag{4.85}$$

and

$$R_i = \alpha_i - \rho_i \rho_i^* \tag{4.86}$$

If we define the jk^{th} element of R_i as $r_{i_{jk}}$, then

$$r_{i_{jk}} = \lambda_{jk} - \lambda_{ij}\,\lambda_{ik}, \; j \neq i,\; k \neq i. \tag{4.87}$$

Now define

$$\hat{\bar{\xi}}_i = \bar{\xi}_i - \dot{m}_i = \bar{\xi}_i - \rho_i\, x. \tag{4.88}$$

Then

$$p\left(\xi_j < x \text{ for all } j \neq i \middle/ \xi_i = x\right)$$

$$= \int_{-\infty}^{x(1-\lambda_{1i})} \int_{-\infty}^{x(1-\lambda_{i-i,\,i})} \int_{-\infty}^{x(1-\lambda_{i+1,\,i})} \int_{-\infty}^{x(1-\lambda_{Mi})} G(\hat{\bar{\xi}}_i; 0; R_i)\, d\left|\hat{\bar{\xi}}_i\right| \tag{4.89}$$

If we now substitute

$$\eta_j = \frac{\hat{\xi}_j}{\sqrt{1-\lambda_{ij}^2}}, \; j = 1, \ldots, i-1, i+1, \ldots, M, \tag{4.90}$$

in the above expression, where $\hat{\xi}_j = \xi_j - \lambda_{ji}\, x$ then

$$E(\eta_j) = 0$$

and

$$E\left[\eta_j\, \eta_k\right] = \frac{\lambda_{jk} - \lambda_{ij}\,\lambda_{ik}}{\sqrt{1-\lambda_{ij}^2}\,\sqrt{1-\lambda_{ik}^2}} \tag{4.91}$$

Therefore

$$p\left(\xi_j < x \text{ for all } j \neq i \middle/ \xi_i = x\right) = \Phi_i\left[x\gamma_{i1}, \ldots, x\gamma_{i,i-1}, x\gamma_{i,i+1}, \ldots, x\gamma_{iM}\right] \tag{4.92}$$

where Φ_i [ ] is an M-1 cumulative Gaussian distribution with zero means, unit variances, and covariances given by

$$\mu_{jk} = \frac{\lambda_{jk} - \lambda_{ij}\lambda_{ik}}{\sqrt{1-\lambda_{ij}^2}\sqrt{1-\lambda_{ik}^2}} \qquad j \neq i, \; k \neq i \tag{4.93}$$

and where

$$\gamma_{ij} = \sqrt{\frac{1-\lambda_{ij}}{1+\lambda_{ij}}} \tag{4.94}$$

Substituting into (4.81), we have

$$P_D(\lambda;\alpha) = \frac{1}{M}\int_{\infty}^{\infty} G(x-\lambda)\,dx \left\{\sum_{i=1}^{M} \Phi_i\left[\gamma x_{i1}, \ldots, x\gamma_{i,i-1}, x\gamma_{i,i+1}, \ldots, x\gamma_{iM}\right]\right\} \tag{4.95}$$

Note that λ appears only in $G(x-\lambda)$. Expanding Φ_i [ ] in a tetra-choric series (summarized in Appendix B) yields

$$\Phi_i\left[x\gamma_{i1}, \ldots, x\gamma_{i,i-1}, x\gamma_{i,i+1}, \ldots, x\gamma_{iM}\right] = \prod_{\substack{j=1 \\ j\neq i}}^{M} \phi(\gamma_{ij}x)$$

$$+ \sum_{r=1}^{\infty} \text{[tetra-choric series]} \tag{4.96}$$

Substituting

$$P_D(\lambda;\alpha) = \frac{1}{M}\int_{-\infty}^{\infty} G(x-\lambda)\left[\sum_{i=1}^{M}\prod_{\substack{j=1 \\ j\neq i}}^{M} \phi(\gamma_{ij}x)\right]dx + \frac{1}{M}R(\lambda;\alpha) \tag{4.97}$$

where

$$R(\lambda;\alpha) = e^{-\frac{1}{2}\lambda^2}\int_{-\infty}^{\infty} e^{\lambda x} e^{-\frac{1}{2}x^2}\,dx \sum_{i=1}^{M}\left\{\sum_{r=1}^{\infty}\text{[tetra-choric series]}\right\} \tag{4.98}$$

The above integral is bounded; therefore $R(\lambda;\alpha)$ is of the order of $e^{-\frac{1}{2}\lambda^2}$ for large λ.

Rewrite the first integral in (4.97) as

$$\frac{1}{M}\int_{-\infty}^{\infty} G(x)\, dx \sum_{i=1}^{M} \prod_{\substack{j=1\\ j\neq i}}^{M} \phi\Big(\gamma_{ij}[x+\lambda]\Big) \tag{4.99}$$

and approximate $\phi(t)$ by

$$1 - \frac{e^{-\frac{1}{2}t^2}}{t\sqrt{2\pi}} \quad \text{for large } t. \tag{4.100}$$

Substituting, for large λ we then have the approximation

$$P_D(\lambda;\alpha) \approx \frac{1}{M}\int_{\infty}^{\infty} G(x)\, dx \sum_{i=1}^{M} \prod_{\substack{j=1\\ j\neq i}}^{M} \left[1 - \frac{e^{-\frac{1}{2}\left[\gamma_{ij}\ (x+\lambda)\right]^2}}{\gamma_{ij}\ (x+\lambda)\ \sqrt{2\pi}}\right] \tag{4.101}$$

For large λ, we can approximate the product expansion by the first two terms, namely

$$\prod_{\substack{j=1\\ j\neq i}}^{M} \left[1 - \frac{e^{-\frac{1}{2}\gamma_{ij}^2\ (x+\lambda)^2}}{\gamma_{ij}(x+\lambda)\ \sqrt{2\pi}}\right] \approx 1 - \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{e^{-\frac{1}{2}\ \gamma_{ij}^2 (x+\lambda)^2}}{\gamma_{ij}(x+\lambda)\sqrt{2\pi}} \tag{4.102}$$

which, when substituted into (4.101) gives

$$P_D\ (\lambda;\alpha) \approx 1 - \frac{1}{M}\sum_{i\neq}^{M}\sum_{j}^{M} \int_{-\infty}^{\infty} G(x)\ dx\ \frac{e^{-\frac{1}{2}\gamma_{ij}^{2}(x+\lambda)^2}}{\gamma_{ij}(x+\lambda)\ \sqrt{2\pi}} \tag{4.103}$$

which, for large λ, can be finally approximated by

$$P_D\ (\lambda;\alpha) \approx 1 - \frac{1}{M}\sum_{i}^{M}\sum_{\neq j}^{M} \frac{e^{-\frac{1}{2}\ \gamma_{ij}^2\ \lambda^2}}{\gamma_{ij}\ \lambda\sqrt{2\pi}} \tag{4.104}$$

QED

Using (4.92) in the previous proof for $P_D(\lambda;\alpha)$, we can obtain a recursion formula for the probability density corresponding to the regular simplex. In this case

$$\lambda_{ij} = \frac{-1}{M-1} \tag{4.105}$$

from which

$$\mu_{jk} = \frac{-1}{M-2} \qquad \text{for all } j \neq k \tag{4.106}$$

which corresponds to the regular simplex in M-2 dimensions.

Also

$$\gamma_{ij} = \sqrt{\frac{M}{M-2}} \tag{4.107}$$

for the regular simplex.

Substituting

$$\Phi_i\left(\gamma_{1i}x, \ldots x\gamma_{i-1,i}, x\gamma_{i+1,i}, \ldots, x\gamma_{Mi}\right)\Bigg|_{\alpha_R} = \underbrace{\int_{-\infty}^{x\sqrt{\frac{M}{M-2}}} \cdots \int}_{M-1 \text{ fold}} p_{M-1}(y;\alpha_R)\, d|y| \tag{4.108}$$

Since

$$p_M(x;\alpha_R) = MG(x)\, \Phi_i\left(\gamma_{1i}x, \ldots, x\gamma_{i-1,i}, x\gamma_{i+1,i}, \ldots, x\gamma_{Mi}\right)\Bigg|_{\alpha_R} \tag{4.109}$$

we obtain the recursion formula

$$p_M(x;\alpha_R) = MG(x) \underbrace{\int_{-\infty}^{x\sqrt{\frac{M}{M-2}}} \cdots \int}_{M-1 \text{ fold}} p_{M-1}(y;\alpha_R)\, d|y| \tag{4.110}$$

By repeated use of this relationship, $P_D(\lambda;\alpha_R)$ can be expressed in terms of successive integrations of $G(x)$. This then can be used for numerical evaluation of $P_D(\lambda;\alpha_R)$.

For any α, we now determine an upper bound on the magnitude of

$$\frac{\partial P_D(\lambda;\alpha)}{\partial \lambda_{ij}}$$

Lemma 4.7:

Define

$$P_{ij}(\lambda;\alpha) = \frac{\partial P_D(\lambda;\alpha)}{\partial \lambda_{ij}} \tag{4.111}$$

Then

$$\left| P_{ij}(\lambda;\alpha) \right| \leq \frac{\lambda e^{-\frac{1}{2}\lambda^2}}{M^2\, 2\sqrt{\pi}\sqrt{1-\lambda_{ij}}}\, e^{\frac{\lambda^2(1+\lambda_{ij})}{4}} \tag{4.112}$$

Proof:

It is sufficient to prove (4.112) for λ_{12}. From (4.57)

$$P_{12}(\lambda;\alpha) = \frac{-\lambda}{M^2} e^{-\frac{1}{2}\lambda^2} \phi_{12}(\lambda;\alpha) \tag{4.113}$$

and from (4.66)

$$\frac{\phi_{12}(\lambda;\alpha)}{M} = \int_0^\infty e^{\lambda x} G_{12}(x) F_{12}(x)\, dx \tag{4.114}$$

where $G_{12}(x)$ is given by (4.63) and $F_{12}(x)$ by (4.69).

For $\phi_{12}(\lambda;\alpha)$, let

$$x = u_{12}\, y \tag{4.115}$$

where

$$u_{12} = \sqrt{\frac{1+\lambda_{12}}{2}} \tag{4.116}$$

Then (4. 114) becomes

$$\frac{\phi_{12}(\lambda;\alpha)}{M} = \frac{1}{2\sqrt{\pi}\,\sqrt{1-\lambda_{12}}}\; e^{\frac{\lambda^2(1+\lambda_{12})}{4}}\; R_{12}(\lambda) \tag{4.117}$$

where

$$R_{12}(\lambda) = \int_0^{\infty} G(y-\lambda\, u_{12})\, F_{12}(yu_{12})\, dy \tag{4.118}$$

Since $F_{12}(\quad)$ is a cumulative density function, it is clear that

$$0 \le R_{12}(\lambda) \le 1.$$

Therefore

$$\left|P_{12}(\lambda;\alpha)\right| \le \frac{\lambda}{M^2}\, e^{-\frac{1}{2}\lambda^2}\; \frac{1}{2\sqrt{\pi}\,\sqrt{1-\lambda_{12}}}\; e^{\frac{\lambda^2(1+\lambda_{12})}{4}} \tag{4.119}$$

QED

As was shown in the special case when D = 2, we now prove the following, also a special case of Shannon's coding theorems, but again by a more direct derivation.

Theorem 4. 7:

For the orthogonal signal structure, as $\lambda \to \infty$ and $M \to \infty$ so that the Communication Efficiency $\beta = \frac{\lambda^2}{\log_2 M}$ (from (3. 24)) remains fixed:

$$\lim_{\substack{\lambda\to\infty \\ M\to\infty}} P_D(\lambda;\alpha_o) = \begin{cases} 1 \text{ if } \beta \log_2 e > 2 \\ 0 \text{ if } \beta \log_2 e < 2 \end{cases} \tag{4.120}$$

Proof: From (4. 28)

$$P_D(\lambda;\alpha_o) = \int_{\infty}^{\infty} G(x) \left[\phi\,(x+\lambda)\right]^{M-1} dx$$

Now

$$\left[\phi(x)\right]^{M-1} = \exp\left\{(M-1)\ \ell n\left[\phi(x)\right]\right\}$$

and

$$\ell n\left[\phi(x)\right] = \ell n\left[1 - \left(1 - \phi(x)\right)\right]$$

But

$$1 - \phi(x) \approx \frac{1}{x\sqrt{2\pi}}\ e^{-\frac{1}{2}x^2} \quad \text{for large x.}$$

Thus

$$\ell n\left[1 - \left(1 - \phi(x)\right)\right] \approx \ell n\left[1 - \frac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}}\right]$$

Since

$$\ell n\ (1 - a) \approx -a \text{ for small a,}$$

we have

$$\ell n\left[\phi(x+\lambda)\right] \approx \frac{-1}{(x+\lambda)\sqrt{2\pi}}\ e^{-\frac{1}{2}(x+\lambda)^2} \approx \frac{-1}{\lambda\sqrt{2\pi}}\ e^{-\frac{1}{2}\lambda^2}$$

for large λ.

Hence, for large λ

$$\left[\phi(x+\lambda)\right]^{M-1} \approx \exp\left\{\frac{-(M-1)}{\lambda\sqrt{2\pi}}\ e^{-\frac{1}{2}\lambda^2}\right\}$$

$$\approx \exp\left\{\frac{-(M-1)}{\sqrt{2\pi}\ \sqrt{\beta\log_2 M}}\ e^{-\frac{1}{2}\beta\log_2 M}\right\}$$

By noting that

$$\log_2 M = \log_2 e\ \ell n\ M$$

we have

$$\left[\phi(x+\lambda)\right]^{M-1} \approx \exp\left\{\frac{-(M-1)\ e^{-\frac{1}{2}\beta\log_2 e\ \ell n\ M}}{\sqrt{2\pi}\sqrt{\log_2 e\ \ell n\ M}}\right\}$$

$$\approx \exp\left\{\frac{-(M-1)(M)^{-\frac{1}{2}\beta\log_2 e}}{\sqrt{2\pi}\sqrt{\log_2 e\ \ell n\ M}}\right\} \qquad (4.121)$$

Thus, as $M \to \infty$, if $\frac{1}{2}\ \beta \log_2 e > 1$, then (4.121) $\to 1$, and $P_D(\lambda \to \infty; \alpha_o) = 1$; and if $\frac{1}{2}\ \beta \log_2 e < 1$, $P_D(\lambda \to \infty; \alpha_o) = 0$. QED

An extension of this result is the following:

Corollary. For each M let

$$\gamma_M = \max_{i \neq j} \lambda_{ij} \tag{4.122}$$

Then as $\lambda \to \infty$ and $M \to \infty$

$$P_D(\lambda;\alpha) \to 1 \tag{4.123}$$

for β such that

$$\lim_{M} \beta\left[1 - \gamma_M\right] > 2 \log_2 e \tag{4.124}$$

Proof: Let

$$\alpha_{\gamma_M} = \begin{pmatrix} 1 & & \gamma_M \\ & \ddots & \\ \gamma_M & & 1 \end{pmatrix} \tag{4.125}$$

Then from Theorem 4.3

$$P_D\ (\lambda;\alpha) \geq P_D\left(\lambda;\alpha_{\gamma_M}\right)$$

and from (4.40)

$$P_D\left(\lambda;\alpha_{\gamma_M}\right) = P_D\left(\lambda\sqrt{1 - \gamma_M};\ \alpha_o\right)$$

or equivalently

$$P_D(\lambda;\alpha) \geq P_D\left(\lambda\sqrt{1 - \gamma_M};\ \alpha_o\right) \tag{4.126}$$

Now let

$$\bar{\lambda} = \lambda\sqrt{1 - \gamma_M} \tag{4.127}$$

and

$$\bar{\beta} = \frac{\bar{\lambda}^2}{\log_2 M} \tag{4.128}$$

Then from above

$$\lim_{\substack{\lambda \to \infty \\ M \to \infty}} P_D (\lambda; \alpha_o) = 1$$

if

$$\bar{\beta} \log_2 e > 2$$

from which we get

$$\lim_{\substack{\lambda \to \infty \\ M \to \infty}} P_D (\lambda; \alpha) = 1 \tag{4.129}$$

if

$$\lim_{M} \beta \sqrt{1 - \gamma_M} > \frac{2}{\log_2 e} \tag{4.130}$$

QED

(4.125) provides an upper bound for the probability of error for any α which can be evaluated numerically.

Again it should be emphasized that these asymptotic results agree with Shannon's limit theorems, but they do not provide us with optimal signal sets.

REFERENCES

4.1 Max, J., "Signals Sets With Uniform Correlation Properties", J. SIAM, 10: 113-118, March, 1962.

4.2 Balakrishnan, A.V., "A Contribution to the Sphere Packing Problem of Communication Theory," Journal of Math. Analysis and Applications, 3: 485-506, December, 1961.

4.3 Slepian, D., "Signaling Systems," Technical Memorandum, Bell Telephone Laboratory, May 7, 1951.

4.4 Bonnesen, T., and Fenchel, W., "Theorie der Konvexen Korper," Chelsea, New York, 1948.

4.5 Hadley, G., "Linear Algebra," Addison Wesley Co., 1961.

4.6 Balakrishnan, A.V., and Taber, J.E., "Error Rates in Coherent Communication Systems," IRE Trans. of P.G. on Communication Systems, Vol. CS-10, No. 1, pp. 86-89, March, 1962.

4.7 Kendall, M.G., "Proof of Relations Connected With Tetra-Choric Series and its Generalizations", Biometrica 32, 196(1941).

V. OPTIMALITY WHEN DIMENSIONALITY IS NOT SPECIFIED; REGULAR SIMPLEX CODING

We now consider the problem of optimal signal design when there is no restriction on the number of degrees of freedom, D. As shown in the previous chapter, the class of α for which D $(\alpha) = 0$ contains all of the optimal α. Also, we concluded that the maximum value D need be is M-1. Therefore, even in this case of no bandwidth restriction, only a finite bandwidth is required for the optimal signal set. This again does not violate Shannon's channel capacity theorem since M and T can vary there, but are fixed and finite here.

5.1 Necessary (First Order) Considerations for Optimality

If we fix λ, then, since $P_D(\lambda;\alpha)$ is a continuous function in α, and since Γ is closed and bounded, it follows that the optimal α is actually attained at some point α_o in Γ. However, α_o may depend on λ.

Since $P_D(\lambda;\alpha)$ is analytic, which results from the fact that it is expandable in a tetra-choric series, it is, of course, differentiable. Therefore, the directional derivative in Γ space, directed away from α_o towards any other admissible α must be nonpositive. Since Γ is convex the adjoining line segment from α_o to α will also be in Γ.

Now, for any other admissible choice, say α', expand $P_D(\lambda;\alpha')$ about $P_D(\lambda;\alpha_o)$ in the Taylor series expansion

$$P_D(\lambda;\alpha') = P_D(\lambda;\alpha_o) + \sum_{i>j}\sum \left(\lambda'_{ij} - \lambda^\circ_{ij}\right) \frac{\partial P_D(\lambda;\alpha_o)}{\partial\lambda_{ij}} + \sum_{i>j}\sum \sum_{k>l}\sum \left(\lambda'_{ij} - \lambda^\circ_{ij}\right)$$

$$\cdot \left(\lambda'_{kl} - \lambda^\circ_{kl}\right) \frac{\partial^2 P_D(\lambda;\alpha_o)}{\partial\lambda_{ij}\,\partial\lambda_{kl}} + \ldots \tag{5.1}$$

If α_o is truly the optimal choice, then the first order term in (5.1) must be nonpositive for any α' in the neighborhood of α_o.

Theorem 5.1. For the regular simplex signal structure, α_R, the first order variation in (5.1) is nonpositive for any admissible α' at all signal-to-noise ratios.

Proof: At α_R, $\frac{\partial P_D(\lambda;\alpha_R)}{\partial\lambda_{ij}}$ is independent of i and j and is strictly negative. Thus it is sufficient to show

$$\sum_{i>}\sum_{j}\left(\lambda'_{ij} - \lambda^R_{ij}\right) \geq 0 \tag{5.2}$$

For this we have only to note that

$$\left(\sum_{i=1}^{M} S'_i\right)^* \left(\sum_{j=1}^{M} S'_j\right) = M + 2\sum_{i>}\sum_{j} \lambda'_{ij} \geq 0 \tag{5.3}$$

or equivalently

$$\sum_{i>}\sum_{j} \lambda'_{ij} \geq -\frac{M}{2} \tag{5.4}$$

However, for α_R

$$\sum_{i>}\sum_{j} \lambda^R_{ij} = -\frac{M}{2} \tag{5.5}$$

which validates (5.2) and proves the theorem. QED

Note, however, that equality exists in (5.2) for any α' such that $\sum_{i=1}^{M} S'_i = 0$. Also, from Lemma 4.5 there is strict inequality in (5.2) for all $\{S'_i\}$ such that $\sum_{i=1}^{M} S'_i \neq 0$, and this class of sets is off the tangent plane defined by

$$\sum_{i=1}^{M}\sum_{j=1}^{M} \lambda_{ij} = 0 \tag{5.6}$$

Therefore, Theorem 5.1 proves that α_R is a local maximum in every admissible direction in Γ except those in the tangent plane given by (5.6). In these directions Theorem 5.1 shows that α_R is a local extremum. To prove that α_R is also a local maximum in these directions, the second order variation in the Taylor series expansion must be examined, which we do in Section 5.4.

Certain properties which α_o must possess are now derived. Consider an α' on the surface of Γ, but not on the tangent plane defined by (5.6), and consider points on the adjoining line segment

$$\alpha = (1-\theta)\,\alpha_o + \theta\,\alpha', \; 0 \leq \theta \leq 1. \tag{5.7}$$

The eigenvector corresponding to zero eigenvalue for α' is not $(1,\ldots,1)^*$ as it is for α_o, since α' is not in the tangent plane defined by (5.6). This statement requires that α_o be in this tangent plane and therefore has eigenvector $(1,\ldots,1)^*$ corresponding to zero eigenvalue which we assume for the present. Therefore, for $0 < \theta < 1$, α is in the interior of Γ, for which $D(\alpha) > 0$, and from Theorem 4.2, α can be projected onto the boundary of Γ at a point $\tilde{\alpha}$ and improve the probability of detection (as shown in Figure 5.1). Now look at $P_D(\lambda;\tilde{\alpha})$ as $\tilde{\alpha} \to \alpha_o$. That is, we are examining the probability of detection along a specific path in the neighborhood of α_o which is entirely on the surface of Γ. The characteristic that α_o must possess which results from this analysis can be stated in the following.

Theorem 5.2: As $\tilde{\alpha} \to \alpha_o$

$$P_D(\lambda;\tilde{\alpha}) \leq P_D(\lambda;\alpha_o) \tag{5.8}$$

for θ sufficiently small, and any α' if

$$P_{ij}(\lambda;\alpha_o) = k\, \overset{\circ}{C}_{ij} \quad \text{for all } i \neq j \tag{5.9}$$

where

$$P_{ij}(\lambda;\alpha_o) = \frac{\partial P_D(\lambda;\alpha_o)}{\partial \lambda_{ij}} \tag{5.10}$$

and $\overset{\circ}{C}_{ij}$ is the cofactor (including sign) of $\overset{\circ}{\lambda}_{ij}$ in α_o.

Proof:

For $0 < \theta < 1$ we have α in the interior of Γ. Define C_{ij} as the cofactor of λ_{ij} in α. For a fixed θ, project α onto the boundary of Γ as in Theorem 4.2, which results in

$$\tilde{\alpha} = \frac{\alpha - \rho^2(1)}{1 - \rho^2} = \frac{(1-\theta)\,\alpha_o + \theta\,\alpha' - \rho^2(1)}{1 - \rho^2} \tag{5.11}$$

where (1) is the M-by-M matrix of all 1's, and from Lemma 4.2

$$\rho^2 = \frac{D}{C} \tag{5.12}$$

where

$$D = D(\alpha)$$

and

$$C = \sum_{i=1}^{M} \sum_{j=1}^{M} C_{ij} \tag{5.13}$$

Then $\tilde{\alpha}$ can be expressed in terms of D and C as

$$\tilde{\alpha} = \frac{C\left[\,(1-\theta)\,\alpha_o + \theta\,\alpha'\,\right] - D(1)}{C-D} \tag{5.14}$$

or

$$\tilde{\lambda}_{ij} = \frac{C\,\lambda_{ij} - D}{C-D} \tag{5.14a}$$

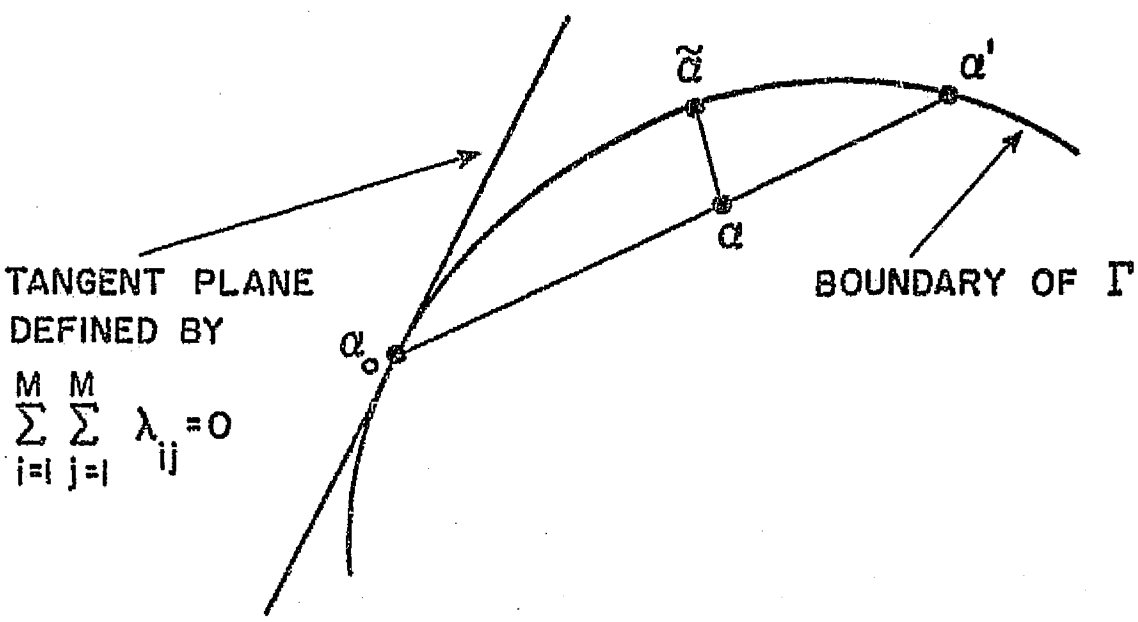


FIGURE 5.1

The directional derivative at α_o along this perscribed path is

$$\left.\frac{\partial P_D(\lambda;\tilde{\alpha})}{\partial\theta}\right|_{\theta=0} = \sum_{i>j}\sum P_{ij}(\lambda;\alpha_o)\left.\frac{\partial\tilde{\lambda}_{ij}}{\partial\theta}\right|_{\theta=0} \tag{5.15}$$

Now

$$\left.\frac{\partial\tilde{\lambda}_{ij}}{\partial\theta}\right|_{\theta=0} = \left(\lambda'_{ij} - \lambda^\circ_{ij}\right) - \frac{1}{C}\left(1 - \lambda^\circ_{ij}\right)\left.\frac{\partial D}{\partial\theta}\right|_{\theta=0} \tag{5.16}$$

and $\left.\frac{\partial D}{\partial\theta}\right|_{\theta=0}$ can be determined by writing

$$D = D(\alpha_o) + \left\{\sum_{i>j}\sum \left.\frac{\partial D(\alpha)}{\partial\lambda_{ij}}\right|_{\theta=0}\left.\frac{\partial\lambda_{ij}}{\partial\theta}\right|_{\theta=0}\right\}\theta + \ldots \tag{5.17}$$

From Lemma 4.6

$$\left.\frac{\partial D(\alpha)}{\partial\lambda_{ij}}\right|_{\theta=0} = \left.2C_{ij}\right|_{\theta=0} = 2C^\circ_{ij} \tag{5.18}$$

Also

$$\frac{\partial\lambda_{ij}}{\partial\theta} = \left(\lambda'_{ij} - \lambda^\circ_{ij}\right) \tag{5.19}$$

from which

$$\left.\frac{\partial D}{\partial\theta}\right|_{\theta=0} = 2\sum_{i>j}\sum\left(\lambda'_{ij} - \lambda^\circ_{ij}\right)C^\circ_{ij} \tag{5.20}$$

Substituting (5.16) and (5.20) into (5.15) we have

$$\left.\frac{\partial P_D(\lambda;\tilde{\alpha})}{\partial\theta}\right|_{\theta=0} = \sum_{i>j}\sum\left(\lambda'_{ij} - \lambda^\circ_{ij}\right)P_{ij}\left(\lambda;\alpha_o\right) - 2\left[\sum_{i>j}\sum\left(\lambda'_{ij} - \lambda^\circ_{ij}\right)C^\circ_{ij}\right] \cdot \left[\frac{\sum_{k>l}\sum(1-\lambda^\circ_{kl})P_{kl}(\lambda;\alpha_o)}{C^\circ}\right] \tag{5.21}$$

For α_o to be a maximum independent of α', (5. 21) must vanish, which occurs only when

$$P_{ij}(\lambda;\alpha_o) = k\, C^{\circ}_{ij} \qquad \text{for all } i \neq j \tag{5. 9}$$

QED

This result can be formulated in another way by use of the Lagrange variational technique. Since we have seen that $D(\alpha_o) = 0$, we form the Lagrangian functional

$$L(\lambda;\alpha) = P_D(\lambda;\alpha) + \nu\, D(\alpha) \tag{5. 22}$$

where ν is the Lagrangian multiplier. Differentiating with respect to λ_{ij} and equating to zero at α_o yields

$$P_{ij}(\lambda;\alpha_o) = k\, C^{\circ}_{ij} \tag{5. 9}$$

as the necessary property α_o must possess to be a local extremum. The regular simplex satisfies (5. 9) since the $C_{ij}\Big|_{\alpha_R}$ are all equal for all $i \neq j$, as also are all the $P_{ij}(\lambda;\alpha_R)$.

Since $C_{ij}\Big|_{\alpha_R} > 0$ and $P_{ij}(\lambda;\alpha_R) < 0$ we require $k < 0$ for α_R to satisfy (5. 9). Note that Theorem 5. 2 required that we choose α' off of the tangent plane defined by

$$\sum_{i=1}\sum_{j=1} \lambda_{ij} = 0$$

Thus, this theorem provides a property which α_o must possess, but still does not answer the question of how $P_D(\lambda;\alpha)$ behaves in the tangent plane in the neighborhood of α_R.

5. 2 Uniqueness of the Regular Simplex Satisfying Necessary Conditions for all SNR

In this section we will show that in all of Γ, α_R is the only signal structure that satisfies the above first order conditions at all signal-to-noise ratios. With this fact, then, if α_R is not the global optimum at all signal-to-noise ratios, then the global optimum, whatever it may be, must necessarily depend in some way on SNR. Or phrased in another way, we can say that with this fact, then the only remaining fact necessary to conclude that α_R is the global maximum at all SNR is that the global optimum itself be independent of SNR. To date, however, this question of the global optimum being independent of SNR is still open.

Before stating and proving this uniqueness theorem, however, it is necessary to first derive several properties that α_o must possess.

To begin, again fix λ, and let α_o correspond to one of the local extrema that is independent of λ (any one if there are several). The relationship in (5. 9) must hold for each of these, or equivalently

$$\phi_{ij}\ (\lambda; \alpha_o) = K\, C^{\circ}_{ij}, \quad i \neq j \tag{5. 23}$$

where $\phi_{ij}\ (\lambda;\alpha)$ is defined by (4. 57) and is non-negative for all $i \neq j$ and any α.

Since the directional derivative of $P_D(\lambda;\alpha_o)$ along any admissible path directed away from α_o must be nonpositive, we have

$$\sum_{i > j}\sum (\lambda_{ij} - \lambda^{\circ}_{ij})\ \phi_{ij}\ (\lambda;\alpha_o) \geq 0 \tag{5. 24}$$

Substituting (5. 23) into (5. 24) gives

$$K \sum_{i > j}\sum\ (\lambda_{ij} - \lambda^{\circ}_{ij})\ C^{\circ}_{ij} \geq 0 \tag{5. 25}$$

Putting in particular

$$\lambda_{ij} = 0, \quad i \neq j \tag{5. 26}$$

which corresponds to the line segment from α_o toward the orthogonal signal set, we get

$$K \sum_{i < j}\sum\ \lambda^{\circ}_{ij}\ C^{\circ}_{ij} \leq 0 \tag{5. 27}$$

With this we have

Lemma 5.1. $$K > 0 \tag{5.28}$$

Proof: We know that $D(\alpha_o) = 0$. Hence

$$M\, D(\alpha_o) = \sum_{i=1}^{M} C_{ii}^{\circ} + \sum_{i\neq}^{M} \sum_{j}^{M} \lambda_{ij}^{\circ}\, C_{ij}^{\circ} = 0$$

α_o is non-negative definite, which implies

$$C_{ii}^{\circ} \geq 0 \qquad i = 1, \ldots, M$$

from which we get

$$\sum_{i \neq} \sum_{j} \lambda_{ij}^{\circ}\, C_{ij}^{\circ} \leq 0$$

Therefore, from (5.27)

$$K \geq 0$$

If $K = 0$, however, then $\phi_{ij}^{\circ} = 0$ for all $i \neq j$, which from Lemma 4.4 implies $\lambda_{ij}^{\circ} = -1$ for all $i \neq j$. This is impossible for $M > 2$. But since the $M = 2$ result is already well known we consider only $M > 2$ and conclude that $K > 0$.

QED

With this Lemma and realizing that $\phi_{ij} \geq 0$ for all $i \neq j$ we have that

$$C_{ij}^{\circ} \geq 0 \text{ for all } i \neq j. \tag{5.29}$$

Actually we can write

Lemma 5.2: For any local extremum in Γ

$$C_{ij}^{\circ} > 0 \text{ for all } i \neq j \tag{5.30}$$

Proof: Suppose $C_{12}^{\circ} = 0$. Then from (5.23) and Lemma 4.4, $\lambda_{12}^{\circ} = -1$. Hence ξ_1 and ξ_2 are linearly dependent and any covariance matrix involving ξ_1 and ξ_2 will have its corresponding determinant equal to zero. Therefore

$$C_{ii}^{\circ} = 0, \quad i = 3, \ldots, M.$$

Now, since α_o is non-negative definite, the matrix of cofactors $\left\{C^{\circ}_{ij}\right\}$ also is non-negative definite, which implies

$$C^{\circ}_{ii}\, C^{\circ}_{jj} - \left(C^{\circ}_{ij}\right)^2 \geq 0.$$

Hence

$$C^{\circ}_{ij} = 0 \text{ for } \begin{array}{l} i,\ j = 3, \ldots, M \\ i = 1,\ j = 3, \ldots, M \\ \text{and } \ i = 2,\ j = 3, \ldots, M; \end{array}$$

which implies

$$D(\alpha_o) = C^{\circ}_{11} + C^{\circ}_{12}\, \lambda^{\circ}_{12} = 0$$

But since we assumed $C^{\circ}_{12} = 0$, we have $C^{\circ}_{11} = 0$. Similarily $C^{\circ}_{22} = 0$.

Therefore

$$C^{\circ}_{ij} = 0 \quad \text{for all } i,\ j$$

and hence

$$\phi^{\circ}_{ij} = 0 \quad \text{for all } i,\ j$$

which implies

$$\lambda^{\circ}_{ij} = -1 \ \text{ for all } i,\ j$$

which is impossible (again for $M > 2$). QED

Actually, we can say more about these cofactors.

Lemma 5.3:

$$C^{\circ}_{ii}\, C^{\circ}_{jj} = \left(C^{\circ}_{ij}\right)^2 \tag{5.31}$$

Proof: Since $D(\alpha_o) = 0$

$$\sum_{i=1}^{M} \lambda^{\circ}_{ij}\, C^{\circ}_{ik} = 0 \quad j, k = 1, \ldots, M.$$

For $j \neq k$, the above is zero for any matrix. For $j = k$, it is zero because $D(\alpha_o) = 0$. Now, if we define C° as the matrix of cofactors $\left\{C^\circ_{ij}\right\}$, then

$$(\alpha_o)(C^\circ) = (0)$$

where (0) is the matrix of all zeros. Now α_o can have rank as high as M-1 and for $\alpha_o = \alpha_R$ the rank is M-1. Thus the rank of C° is at most 1, implying all 2-by-2 minors of C° are zero.

Therefore

$$C^\circ_{ii}\ C^\circ_{jj} = \left(C^\circ_{ij}\right)^2$$

QED

Another significant property of α_o can be stated in the following.

Lemma 5.4: For α_o, the flat in which the probability density is concentrated is given by

$$\sum_{i=1}^{M} \sqrt{C^\circ_{ii}}\ \xi_i = 0$$

Proof: It is immediate that

$$E\left(\sum_{i=1}^{M} \sqrt{C^\circ_{ii}}\ \xi_i\right) = 0$$

Also

$$E\left(\left[\sum_{i=1}^{M} \sqrt{C^\circ_{ii}}\ \xi_i\right]^2\right) = \sum_{i=1}^{M}\sum_{j=1}^{M} \sqrt{C^\circ_{ii}}\sqrt{C^\circ_{jj}}\ \lambda^\circ_{ij}$$

Applying Lemma 5.3, and using the fact that $D(\alpha_o) = 0$, we conclude that the variance of

$$\sum_{i=1}^{M} \sqrt{C^\circ_{ii}}\ \xi_i$$

is zero. Hence it is zero with probability one. QED

We now state and prove the main result of this section.

Theorem 5.3: The regular simplex signal structure is the only signal set which is a local extremum in the class of all admissible signal sets, Γ, at all signal-to-noise ratios.

Proof: For any of the local extremum, from (5.9) we have

$$\phi_{ij}(\lambda;\alpha_o) = K\, C^{\circ}_{ij} = \int_o^{\infty} e^{\lambda x}\, G_{ij}(x)\, F_{ij}(x)\, dx \tag{5.32}$$

where $F_{ij}(x)$ is given by (4.67) and $G_{ij}(x)$ by (4.63). Using this we obtain that

$$K \int_o^{\infty} e^{\lambda x} \left[C^{\circ}_{ij}\, G_{kl}(x)\, F_{kl}(x) - C^{\circ}_{kl}\, G_{ij}(x)\, F_{ij}(x) \right] dx = 0 \tag{5.33}$$

From Lemma 5.3, $K > 0$, hence the integral must vanish. The integral is over $(0, \infty)$ instead of $(-\infty, \infty)$ as a result of Theorems 4.4 and 4.5. Because of the uniqueness and analytic properties of the Laplace transform, equality can be attained in (5.33) if and only if

$$C^{\circ}_{ij}\, G_{kl}(x)\, F_{kl}(x) = C^{\circ}_{kl}\, G_{ij}(x)\, F_{ij}(x) \qquad \text{for all } x. \tag{5.34}$$

Note that we are able to associate the integral in (5.32) with the Laplace transform and conclude that (5.34) must be satisfied only if the assumption is made that $G_{ij}(x)$ and $F_{ij}(x)$ are independent of λ or equivalently that α_o is independent of λ.

It is immediate that α_R satisfies (5.34) since

$$C_{ij}\Big|_{\alpha_R} = C_{kl}\Big|_{\alpha_R}$$

$$G_{ij}(x)\Big|_{\alpha_R} = G_{kl}(x)\Big|_{\alpha_R}$$

and

$$F_{ij}(x)\Big|_{\alpha_R} = F_{kl}(x)\Big|_{\alpha_R} \qquad \text{for all } x. \tag{5.35}$$

To prove that α_R is the only signal structure which satisfies (5.34) for all x, suppose $\alpha_o \neq \alpha_R$ also satisfies (5.34) and let

$$\lambda_{12}^{\circ} = \max_{i \neq j} \lambda_{ij}^{\circ} \tag{5.36}$$

Since $\alpha_o \neq \alpha_R$, $\lambda_{12}^{\circ} > \lambda_{kl}^{\circ}$ for some k and l. This implies that for $F_{12}(x)$ as expressed in (4.69)

$$\alpha_j > 0 \qquad j = 3, \ldots, M$$

and therefore

$$\lim_{x \to \infty} F_{12}(x) = 1. \tag{5.37}$$

Now if

$$\lim_{x \to \infty} F_{kl}(x) = a > 0, \tag{5.38}$$

by using the tetra-choric series it can be shown that for large x, $F_{kl}(x)$ is of the form

$$a - x^{-d} e^{-bx^2} \tag{5.39}$$

where $d > 0$ and $b > 0$. Using the same approximation for $F_{12}(x)$ for large x, and substituting into (5.34) we require

$$C_{kl}^{\circ} \left[\frac{e^{-x^2/1+\lambda_{12}^{\circ}}}{2\pi\sqrt{1-\left(\lambda_{12}^{\circ}\right)^2}} \right] \left[a - x^{-d} e^{-bx^2} \right] \approx C_{12}^{\circ} \left[\frac{e^{-x^2/1+\lambda_{kl}^{\circ}}}{2\pi\sqrt{1-\left(\lambda_{kl}^{\circ}\right)^2}} \right] \left[1 - x^{-d'} e^{-b'x^2} \right] \tag{5.40}$$

which for large x is of the form

$$d_1 e^{-x^2/1+\lambda_{12}^{\circ}} \approx d_2 e^{-x^2/1+\lambda_{kl}^{\circ}} \tag{5.41}$$

and it is immediate that equality can be attained only if

$$\lambda_{12}^{\circ} = \lambda_{kl}^{\circ} \tag{5.42}$$

If on the other hand

$$\lim_{x \to \infty} F_{kl}(x) = 0 \tag{5.43}$$

then

$$F_{kl}(x) = o\left[x^{-d} e^{-bx^2}\right]$$

and no value for λ_{kl} will give equality for large x in (5. 34). Hence all the λ°_{ij} must be equal. QED

To this point we have that the regular simplex is a local extremum at every signal-to-noise ratio, and is the only such signal structure in Γ. But there remains the major problem of showing whether it is the global or absolute maximum, and moreover, that the global optimum is independent of signal-to-noise ratio. So far we have that if there exists a global maximum independent of λ, it necessarily must be the regular simplex.

5.3 Global Optimality of the Regular Simplex for Large SNR

We now prove the following.

Theorem 5.4: In the class of all admissible signal sets, Γ, the regular simplex signal structure is the global optimum for large signal-to-noise ratio.

Proof: By substituting (4. 117) into (4. 113), we obtain for any α

$$-\frac{M}{\lambda} e^{\lambda^2/2} P_{12}(\lambda;\alpha) = \frac{1}{2\sqrt{\pi}\sqrt{1-\lambda_{12}}} e^{\frac{\lambda^2(1+\lambda_{12})}{4}} R_{12}(\lambda) \tag{5.45}$$

where $R_{12}(\lambda)$ is defined by (4. 118). Consider now a sequence $\{\lambda_n\}$ such that

$$\lambda_n \to \infty$$

and define

$$\alpha_n \sim \left\{\lambda^n_{ij}\right\} \tag{5.46}$$

as the global optimum at λ_n. At each n let us renumber so that

$$\lambda_{12}^{n} = \max_{i \neq j} \lambda_{ij}^{n} \tag{5.47}$$

Then from (5.37)

$$F_{12}(x) \to 1 \quad \text{as } x \to \infty$$

for each n, and consequently

$$R_{12}(x) \to 1 \quad \text{as } \lambda_n \to \infty \tag{5.48}$$

But for the optimal choice, we already require that

$$\left.\frac{\partial P_D(\lambda_n ; \alpha_\theta)}{\partial \theta}\right|_{\theta=0} = \sum_{i>j}\sum \left(\lambda'_{ij} - \lambda^n_{ij}\right) P_{ij}\left(\lambda_n ; \alpha_n\right) \leq 0 \tag{5.49}$$

for any other admissible α' where

$$\alpha_\theta = (1-\theta)\,\alpha_n + \theta\,\alpha'$$

Substituting (5.45) into (5.49), we require that α_n satisfy

$$\sum_{i>j}\sum \frac{\left(\lambda'_{ij} - \lambda^n_{ij}\right)}{\sqrt{1-\lambda^n_{ij}}}\; e^{\frac{\left(\lambda_n^2\ \lambda_{ij}^n\right)}{4}} \qquad R_{ij}(\lambda_n) \geq 0 \tag{5.50}$$

for any admissible $\left\{\lambda'_{ij}\right\}$ and every n. For n sufficiently large, however,

$$\lambda^{\circ}_{ij} = \text{constant} \quad i \neq j$$

is the only signal structure which satisfies (5.50) for all admissible α'. For if $\alpha_n = \alpha_R$, then by dividing through by

$$e^{\lambda_n^2\ \lambda_{12}^R}$$

and letting n become large, (5.50) reduces to

$$\sum_{i>j}\sum \left(\lambda'_{ij} - \lambda^R_{ij}\right) \geq 0 \tag{5.51}$$

which we already know to be the case.

If, however, we assume that the global maximum is something other than α_R, and again divide through by

$$e^{\lambda_n^2} \lambda_{12}^n$$

and allow n to become sufficiently large, (5.50) now reduces to

$$\sum \sum \frac{\left(\lambda'_{ij} - \lambda_{ij}^n\right)}{\sqrt{1 - \lambda_{ij}^n}} \geq 0 \tag{5.52}$$

where the sum is now only over those (i, j) for which

$$\lambda_{ij}^n = \max_{i \neq j} \lambda_{ij}^n = \lambda_{12}^n.$$

This inequality will not be satisfied for arbitrary admissible α' unless the sum is over all $i \neq j$, from which we conclude that only α_R satisfies (5.50) for large λ. <u>QED</u>

A different proof of this same theorem has recently been given by Ziv (see Reference 5.2).

5.4 Sufficient (Second Order) Conditions for Optimality

As indicated in Section 5.1, a study of first order variations was not enough to conclude that the regular simplex is a local maximum in all admissible directions of Γ. These first order variations about the regular simplex are identically zero when the direction is in the tangent plane defined by

$$\sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_{ij} = 0 \tag{5.9}$$

Hence, the possibility exists that the regular simplex is a saddle point. In order to conclude that α_R indeed is a local maximum in these directions also, it is necessary to examine the second order variations in the neighborhood of the regular simplex. These are obtained by considering the following:

Let α' be any other admissible choice such that

$$||\alpha' - \alpha_R|| < \delta \tag{5.53}$$

for some fixed $\delta > 0$. Let

$$\lambda_{ij}(\theta) = (1-\theta)\,\lambda^R_{ij} + \theta\lambda'_{ij}. \tag{5.54}$$

Then the Taylor series expansion in θ about α_R for the cummulative density function, $\Phi(x;\alpha')$, is

$$\Phi(x;\alpha') = \Phi\left(x;\alpha_R\right) + a_1(x)\,\theta + a_2(x)\,\frac{\theta^2}{2} + \sum_{k=3}^{\infty} a_k(x)\,\frac{\theta^k}{k!} \tag{5.55}$$

where

$$a_1(x) = \sum_{i>j}\sum \frac{\partial\Phi\left(x;\alpha_R\right)}{\partial\lambda_{ij}}\left(\lambda'_{ij} - \lambda^R_{ij}\right) \tag{5.56}$$

and

$$a_2(x) = \sum_{i>j}\sum\ \sum_{k>l}\sum a_{ijkl}\left(\lambda'_{ij} - \lambda^R_{ij}\right)\left(\lambda'_{kl} - \lambda^R_{kl}\right) \tag{5.57}$$

where for convenience we adopt the notation

$$a_{ijkl} = \frac{\partial^2\Phi\left(x;\alpha_R\right)}{\partial\lambda_{ij}\,\partial\lambda_{kl}} \tag{5.58}$$

In order to proceed further, a_{ijkl} must be partially computed, which we carry out in the following three lemmas. The method used in this computation consists in first considering

$$\alpha_\rho = \begin{pmatrix} 1 & & \rho \\ & \ddots & \\ \rho & & 1 \end{pmatrix} \tag{5.59}$$

and then taking limits as

$$\rho \to \frac{-1}{M-1} \tag{5.60}$$

Lemma 5.5. For $i \neq j \neq k \neq l$, define

$$r = a_{ijkl} = a_{1234}. \tag{5.61}$$

Then

$$r = \underbrace{\int_{-\infty}^{x} \cdots \int}_{M-4 \text{ fold}} G\left(x, x, x, x, \xi_5, \ldots, \xi_M; 0; \alpha_R\right) d\xi_5 \ldots d\xi_M = G_4(x) F_4(x) \tag{5.62}$$

where

$$G_4(x) = \left(\frac{1}{2\pi}\right)^2 \sqrt{\frac{(M-1)^4}{M^3 \ (M-4)}} \quad \exp\left\{-2\left(\frac{M-1}{M-4}\right) \ x^2\right\} \tag{5.63}$$

and

$$F_4(x) = \Pr\left\{\psi_i \leq x\sqrt{\frac{M(M-1)}{(M-4) \ (M-5)}} \ ; i = 5, \ldots, M\right\} \tag{5.64}$$

where the ψ_i are Gaussian with zero means, unit variances, and

$$E(\psi_i \psi_j) = \frac{-1}{M-5}, \quad i \neq j. \tag{5.65}$$

Proof: On account of the symmetry of α_R it is immediate that for any $i \neq j \neq k \neq l$

$$a_{ijkl} = a_{1234}.$$

By the same technique used in Section 4.3, it can be shown that

$$\frac{\partial^2 \Phi(x;\alpha_\rho)}{\partial \lambda_{12} \partial \lambda_{13}} = \underbrace{\int_{-\infty}^{x} \cdots \int}_{M-4 \text{ fold}} G\left(x, x, x, x, \xi_5, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_5 \ldots d\xi_M$$

$$= G\left(\xi_1 = x, \xi_2 = x, \xi_3 = x, \xi_4 = x\right) \int_{-\infty}^{x} \cdots \int G\left(\xi_5, \ldots, \xi_M / \xi_1 = x, \xi_2 = x, \xi_3 = x, \xi_4 = x\right) d\xi_5 \ldots d\xi_M$$

$G\left(\xi_1, \xi_2, \xi_3, \xi_4\right)$ is a nonsingular density and can therefore be written down explicitly. If this is done, then, after taking limits

$$G_4(x) = G\left(\xi_1=x,\ \xi_2=x,\ \xi_3=x,\ \xi_4=x\right) = \left(\frac{1}{2\pi}\right)^2 \sqrt{\frac{(M-1)^4}{M^3(M-4)}}\ \exp\left[-2\left(\frac{M-1}{M-4}\right)x^2\right]$$

Hence

$$F_4(x) = \int_{-\infty}^{x}\!\!\cdots\!\int G\left(\xi_5, \ldots, \xi_M \Big/ \xi_1=x,\ \xi_2=x,\ \xi_3=x,\ \xi_4=x\right)\ d\xi_5 \ldots d\xi_M$$

which can be written

$$F_4(x) = \underbrace{\int_{-\infty}^{x}\!\!\cdots\!\int}_{M-4\text{ fold}} G\left(\xi_5, \ldots, \xi_M;\ m_4;\ C_4\right)\ d\xi_5 \ldots d\xi_M$$

where

$$m_4 = \begin{pmatrix} \frac{-4x}{M-4} \\ \vdots \\ \frac{-4x}{M-4} \end{pmatrix}$$

and the elements of C_4 are

$$\frac{M(M-5)}{(M-1)\ (M-4)} \qquad \text{along the diagonal}$$

and

$$\frac{-M}{(M-1)\ (M-4)} \qquad \text{off the diagonal.}$$

Finally, express $F_4(x)$ in terms of normalized random variables by setting

$$\psi_i = \frac{\xi_i + \left(\frac{4x}{M-4}\right)}{\sqrt{\frac{M(M-5)}{(M-4)\ (M-1)}}}, \qquad i = 5, \ldots, M.$$

The ψ_i have zero mean, unit variance, and covariances equal to to $\frac{-1}{(M-5)}$, from which

$$F_4(x) = \Pr\left\{ \psi_i \leq x\sqrt{\frac{M(M-1)}{(M-4)\ (M-5)}}\ ;\ i = 5, \ldots, M \right\}$$

QED

Lemma 5.6. For $j = k$, $j \neq 1$, define

$$q = a_{kjkl} = a_{1213}. \tag{5.66}$$

Then

$$q = G_4(x)\,F_4(x) - \left(\frac{M-1}{M-3}\right)\; x\, G_3(x)\,F_3(x) \tag{5.67}$$

where

$$G_3(x) = \left(\frac{1}{2\pi}\right)^{3/2} \sqrt{\frac{(M-1)^3}{M^2\,(M-3)}} \quad \exp\left\{-\frac{3}{2}\left(\frac{M-1}{M-3}\right) x^2\right\} \tag{5.68}$$

and

$$F_3(x) = \Pr\left\{\psi_i \leq x\sqrt{\frac{M(M-1)}{(M-3)\,(M-4)}}\;\;;\; i = 4, \ldots, M\right\} \tag{5.69}$$

where the ψ_i are Gaussian with zero mean, unit variance, and

$$E\,(\psi_i\,\psi_j) = \frac{-1}{M-4} \quad ,\; i \neq j \tag{5.70}$$

Proof. Again, on account of the symmetry in α_ρ and α_R

$$a_{kjkl} = a_{1213} \qquad \text{for } j \neq k \neq 1.$$

Using the same technique as above

$$\frac{\partial^2 \Phi(x;\alpha_R)}{\partial\lambda_{12}\,\partial\lambda_{13}} = \lim_{\rho\to\frac{-1}{M-1}} \underbrace{\int_{-\infty}^{x}\!\!\cdots\!\int}_{M\text{-fold}} \frac{\partial^4}{\partial\xi_1^2\,\partial\xi_2\,\partial\xi_3}\, G\left(\xi;0;\alpha_\rho\right)\; d|\xi|$$

$$= -\lim_{\rho\to\frac{-1}{M-1}} \underbrace{\int_{-\infty}^{x}\!\!\cdots\!\int}_{M-3\text{ fold}} \frac{\partial^3}{\partial\xi_1\,\partial\xi_2\,\partial\xi_3} \left[\frac{\sum_{i=1}^{M} C_{1i}\xi_i}{D}\right] G\left(\xi;0;\alpha_\rho\right)\; d|\xi|$$

$$= -\lim_{\rho\to\frac{-1}{M-1}} \underbrace{\int_{-\infty}^{x}\!\!\cdots\!\int}_{M-3\text{ fold}} \left[\frac{(1+2\epsilon)x}{\sigma_1^2} + \frac{\epsilon}{\sigma_1^2}\sum_{i=4}^{M}\xi_i\right] G\left(x, x, x, \xi_4, \ldots, \xi_M;0;\alpha_\rho\right) d\xi_4 \ldots d\xi_M$$

where

$$\epsilon = \frac{C_{1i}}{C_{11}} = \frac{-\rho}{[1+(M-2)\rho]}, \quad i \geqq 2,$$

and

$$\sigma_1^2 = \frac{D}{C_{11}} = \frac{(1-\rho)[1+(M-1)\rho]}{[1+(M-2)\rho]}$$

Following the same procedure as in the previous lemma, it can be shown that

$$\lim_{\rho \to \frac{-1}{M-1}} \int_{-\infty}^{x} \cdots \int G\left(x, x, x, \xi_4, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_4 \ldots d\xi_M = G_3(x)\, F_3(x).$$

Substituting

$$q = -\lim_{\rho \to \frac{-1}{M-1}} \left\{ \frac{(1+2\epsilon)x}{\sigma_1^2} \; G_3(x)\, F_3(x) \right.$$

$$\left. + \frac{\epsilon}{\sigma_1^2} \sum_{j=4}^{M} \underbrace{\int_{-\infty}^{x} \cdots \int}_{M-3\text{ fold}} \xi_j\, G\left(x, x, x, \xi_4, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_4 \ldots d\xi_M \right\} \tag{5.71}$$

Now define

$$a_3 = \underbrace{\int_{-\infty}^{x} \cdots \int}_{M-3\text{ fold}} \xi_j\, G\left(x, x, x, \xi_4, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_4 \ldots d\xi_M, \quad j = 4, \ldots, M. \tag{5.72}$$

Here also, a_3 does not depend on j on account of the symmetry of α_ρ. Next we note that

$$G_4(x)\,F_4(x) = \underbrace{\int_{-\infty}^{x}\cdots\int}_{M-4\text{ fold}} G\left(x, x, x, x, \xi_5, \ldots, \xi_M; 0; \alpha_R\right)\, d\xi_5 \ldots d\xi_M$$

$$= \underbrace{\int_{-\infty}^{x}\int}_{M-3\text{ fold}} \frac{\partial}{\partial \xi_4}\, G\left(x, x, x, \xi_4, \ldots, \xi_M; 0; \alpha_R\right)\, d\xi_4 \ldots d\xi_M$$

$$= -\lim_{\rho \to \frac{-1}{M-1}} \int_{-\infty}^{x}\int \left[\frac{3\epsilon x}{\sigma_1^2} + \frac{\sum_{j-4}^{M} C_{4j}\xi_j}{D}\right] G\left(x, x, x, \xi_4, \ldots, \xi_M; 0; \alpha_\rho\right)\, d\xi_4 \ldots d\xi_M$$

$$= -\lim_{\rho \to \frac{-1}{M-1}} \left\{\frac{3\epsilon x}{\sigma_1^2}\, G_3(x)\, F_3(x) + \frac{[1 + (M-3)\,\epsilon]}{\sigma_1^2}\, a_3\right\}$$

from which

$$a_3 = -\lim_{\rho \to \frac{-1}{M-1}} \frac{\sigma_1^2\, G_4(x)\, F_4(x) + 3\epsilon x\, G_3(x)\, F_3(x)}{[1 + (M-4)\,\epsilon]} \tag{5. 73}$$

Substituting into (5. 71)

$$q = \lim_{\rho \to \frac{-1}{M-1}} \frac{(M-3)\,\epsilon}{[1 + (M-1)\,\epsilon]}\, G_4(x)\, F_4(x) - \left\{\frac{1+2\epsilon}{\sigma_1^2} - \frac{3\epsilon^2(M-3)}{[1 + (M-4)\,\epsilon]\,\sigma_1^2}\right\} x\, G_3(x)\, F_3(x)$$

$$= G_4(x)\, F_4(x) - \left(\frac{M-1}{M-3}\right) x\, G_3(x)\, F_3(x)$$

QED

Lemma 5. 7. For $i = k$, and $j = 1$, define

$$p = a_{klkl} = a_{1212}. \tag{5. 74}$$

Then

$$p = \left(\frac{M-1}{M-2}\right)^2 x^2\, G_2(x)\, F_2(x) + \left[C(x) - 2F_2(x)\right] \left[\frac{M-1}{M(M-2)}\right] G_2(x)$$

$$- 2\left(\frac{M-1}{M-2}\right) x\, G_3(x)\, F_3(x) + G_4(x)\, F_4(x) \tag{5. 75}$$

where

$$G_2(x) = \frac{1}{2\pi}\sqrt{\frac{(M-1)^2}{M(M-2)}} \quad \exp\left[-\left(\frac{M-1}{M-2}\right) x^2\right] \tag{5.76}$$

$$F_2(x) = \Pr\left\{\psi_i \le x\beta;\ i = 3, \ldots, M\right\} \tag{5.77}$$

and

$$C(x) = \underbrace{\int_{-\infty}^{x\beta}\!\!\cdots\!\int}_{M-2\text{ fold}} \psi_i^2\ G\left(\psi_3, \ldots, \psi_M; 0; C_3\right)\ d\psi_3 \ldots d\psi_M \qquad i = 3, \ldots, M, \tag{5.78}$$

where

$$\beta = \sqrt{\frac{M(M-1)}{(M-2)\ (M-3)}}$$

and

$$C_3 = \begin{pmatrix} 1 & & \frac{-1}{M-3} \\ & \ddots & \\ \frac{-1}{M-3} & & 1 \end{pmatrix} \tag{5.80}$$

<u>Proof.</u> In the same manner as in the previous lemmas, it can be shown that

$$G_2(x)\ F_2(x) = \lim_{\rho \to \frac{-1}{M-1}} \int_{-\infty}^{x}\!\!\cdots\!\int G\left(x, x, \xi_3, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_3 \ldots d\xi_M \tag{5.81}$$

Now

$$p = \frac{\partial^2 \Phi\ (x; \alpha_R)}{\partial \lambda^2_{12}}$$

$$= \lim_{\rho \to \frac{-1}{M-1}} \int_{-\infty}^{x}\!\!\cdots\!\int \left\{ \frac{\left[(1+\epsilon)x + \epsilon \sum_3^M \xi_i\right]^2}{\sigma_1^4} - \frac{\epsilon}{\sigma_1^2} \right\} G\left(x, x, \xi_3, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_3 \ldots d\xi_M$$

$$= \lim_{\rho \to \frac{-1}{M-1}} G_2(x) \int_{-\infty}^{x}\!\!\cdots\!\int \left\{ \frac{\left[(1+\epsilon)x + \epsilon \sum_3^M \xi_i\right]^2}{\sigma_1^4} - \frac{\epsilon}{\sigma_1^2} \right\} G\left(\xi_3, \ldots, \xi_M \big/ \xi_1 = x, \xi_2 = x\right)$$

$$d\xi_3 \ldots d\xi_M \tag{5.82}$$

Note that

$$G\left(\xi_3, \ldots, \xi_M \big/ \xi_1 = x,\ \xi_2 = x\right) = G\left(\xi_3, \ldots, \xi_M;\ m_2;\ C_2\right)$$

Where

$$m_2 = \begin{pmatrix} \frac{2\rho x}{1+\rho} \\ \vdots \\ \frac{2\rho x}{1+\rho} \end{pmatrix}$$

and the elements of C_2 are

$$1 - \frac{2\rho^2}{1+\rho} \qquad \text{along the diagonal}$$

and

$$\rho - \frac{2\rho^2}{1+\rho} \quad \text{off the diagonal.}$$

Let

$$\hat{\xi}_i = \xi_i - \frac{2\rho}{1+\rho} x, \ i = 3, \ldots, M.$$

Then

$$p = \lim_{\rho \to \frac{-1}{M-1}} G_2(x) \int_{-\infty}^{\gamma x} \cdots \int \left(\frac{\left\{ \left[(1+\epsilon) + \frac{2\rho(M-2)\epsilon}{1+\rho} \right] x \right\}^2}{\sigma_1^4} \right.$$

$$\left. + \frac{\left[(1+\epsilon) + \frac{2\rho(M-2)\epsilon}{1+\rho} \right]}{\sigma_1^4} \epsilon x \left[\sum_{i=3}^{M} \hat{\xi}_i \right] + \frac{\epsilon^2}{\sigma_1^4} \left[\sum_{i=3}^{M} \hat{\xi}_i \right]^2 - \frac{\epsilon}{\sigma_1^2} \right)$$

$$G\left(\hat{\xi}_3, \ldots, \hat{\xi}_M;\ 0;\ C_2\right)\ d\hat{\xi}_3 \ldots d\hat{\xi}_M$$

where

$$\gamma = \frac{1-\rho}{1+\rho}$$

This can be reduced to

$$p = \left(\frac{M-1}{M-2}\right)^2 x^2\, G_2(x)\, F_2(x) + \lim_{\rho \to \frac{-1}{M-1}} G_2(x) \underbrace{\int_{-\infty}^{\gamma x}\!\!\cdots\!\!\int}_{M-2\text{ fold}}$$

$$\left\{ \frac{2\epsilon x}{(1+\rho)\sigma_1^2} \left[\sum_{i=3}^{M} \hat{\xi}_i\right] + \frac{\epsilon}{\sigma_1^4}\left[\sum_{i=3}^{M} \hat{\xi}_i\right]^2 - \frac{\epsilon}{\sigma_1^2} \right\} G\left(\hat{\xi}_3, \ldots, \hat{\xi}_M; 0; C_2\right) d\hat{\xi}_3 \ldots d\hat{\xi}_M$$

(5. 83)

Now define

$$\hat{a} = G_2(x) \int_{-\infty}^{\gamma x}\!\!\cdots\!\!\int \hat{\xi}_i\, G\left(\hat{\xi}_3, \ldots, \hat{\xi}_M; 0; C_2\right) d\hat{\xi}_3 \ldots d\hat{\xi}_M$$

$i = 3, \ldots, M.$ (5. 84)

In the same manner as in the previous lemmas, $\hat{a}$ can be evaluated by noting that

$$G_3(x)\, F_3(x) = \frac{-2\epsilon x}{\sigma_1^2}\, G_2(x)\, F_2(x)$$

$$- \frac{[1+(M-3)\epsilon]}{\sigma_1^2} \underbrace{\int_{-\infty}^{x}\!\!\cdots\!\!\int}_{M-2\text{ fold}} \xi_i\, G\left(x, x, \xi_3, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_3 \ldots d\xi_M \qquad (5.85)$$

and

$$\int_{-\infty}^{x}\!\!\cdots\!\!\int \xi_i\, G\left(x, x, \xi_3, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_3 \ldots d\xi_M$$

$$= G_2(x) \int_{-\infty}^{\gamma x}\!\!\cdots\!\!\int \left[\hat{\xi}_i + \frac{2\rho x}{1+\rho}\right] G\left(\hat{\xi}_3, \ldots, \hat{\xi}_M; 0; C_2\right) d\hat{\xi}_3 \ldots d\hat{\xi}_M \qquad (5.86)$$

Substituting (5. 86) into (5. 85), we obtain that

$$\hat{a} = \left[- \frac{\sigma_1^2}{[1 + (M-3)\epsilon]}\right] G_3(x)\, F_3(x)$$

which when substituted into (5.83) yields

$$p = \left(\frac{M-1}{M-2}\right)^2 x^2 G_2(x) F_2(x) - 2 \left(\frac{M-1}{M-2}\right) x G_3(x) F_3(x) + \lim_{\rho \to \frac{-1}{M-1}} \left\{ \frac{\epsilon}{\sigma_1^4} \left[(M-2) \hat{C} + (M-2)(M-3) \hat{B} \right] - \frac{\epsilon}{\sigma_1^2} G_2(x) F_2(x) \right\} \quad (5.87)$$

where

$$\hat{C} = G_2(x) \int_{-\infty}^{\gamma x} \cdots \int \left(\hat{\xi}_i \right)^2 G\left(\hat{\xi}_3, \ldots, \hat{\xi}_M; 0; C_2 \right) d\hat{\xi}_3 \ldots d\hat{\xi}_M \qquad i = 3, \ldots, M \quad (5.88)$$

and

$$\hat{B} = G_2(x) \int_{-\infty}^{\gamma x} \cdots \int \hat{\xi}_i \hat{\xi}_j G\left(\xi_3, \ldots, \xi_M; 0; C_2 \right) d\hat{\xi}_3 \ldots d\hat{\xi}_M \qquad i \neq j;\ (i, j) \geqq 3 \quad (5.89)$$

Define

$$a_2 = \int_{-\infty}^{x} \cdots \int \xi_i G\left(x, x, \xi_3, \ldots, \xi_M; 0; \alpha_\rho \right) d\xi_3 \ldots d\xi_M \quad (5.90)$$

which can be shown to be equal to

$$a_2 = - \frac{\sigma_1^2 G_3(x) F_3(x) + 2\epsilon x G_2(x) F_2(x)}{[1 + (M-3)\epsilon]} \quad (5.91)$$

Now

$$G_4(x) F_4(x) = \lim_{\rho \to \frac{-1}{M-1}} \int_{-\infty}^{x} \cdots \int \frac{\partial^2}{\partial \xi_3 \partial \xi_4} G\left(x, x, \xi_3, \ldots, \xi_M\ 0; \alpha_\rho \right) d\xi_3 \ldots d\xi_M$$

$$= \lim_{\rho \to \frac{-1}{M-1}} \left[\frac{4\epsilon^2 x^2}{\sigma_1^4} - \frac{\epsilon}{\sigma_1^2} \right] G_2(x) F_2(x) + \frac{4\epsilon x}{\sigma_1^4} \left[1 + (M-3)\epsilon \right] a_2$$

$$+ \frac{1}{\sigma_1^4} \left\{ \left[2\epsilon + (M-4)\epsilon^2 \right] C' + \left[1 + 2(M-4)\epsilon + \left\{ (M-4)(M-3) + 1 \right\} \epsilon^2 \right] B' \right\} \quad (5.92)$$

where

$$C'(x) = \int_{-\infty}^{x}\cdots\int \xi_i^2\, G\left(x, x, \xi_3, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_3 \ldots d\xi_M \quad i = 3, \ldots, M \tag{5.93}$$

and

$$B'(x) = \int_{-\infty}^{x}\cdots\int \xi_i\, \xi_j\, G\left(x, x, \xi_3, \ldots, \xi_M; 0; \alpha_\rho\right) d\xi_3 \ldots d\xi_M$$

$$i \neq j;\ (i, j) \geqq 3. \tag{5.94}$$

By writing $C'(x)$ and $B'(x)$ in terms of $G_2(x)$ and substituting

$$\hat{\xi}_i = \xi_i - \frac{2\rho x}{1+\rho},\ i = 3, \ldots, M$$

we obtain

$$B'(x) = \hat{B}(x) - \frac{4\rho\sigma_1^2 x}{[1+(M-3)\epsilon]\ (1+\rho)}\ G_3(x)\, F_3(x) + \frac{4\rho^2 x^2}{(1+\rho)^2}\ G_2(x)\, F_2(x) \tag{5.95}$$

and

$$C'(x) = \left[1 - \frac{2\rho^2}{1+\rho}\right] G_2(x)\, C(x) - \frac{4\rho\sigma_1^2 x}{[1+(M-3)\epsilon]\ (1+\rho)}\ G_3(x)\, F_3(x)$$
$$+ \frac{4\rho^2 x^2}{(1+\rho)^2}\ G_2(x)\, F_2(x). \tag{5.96}$$

Also

$$\hat{C}(x) = \left[1 - \frac{2\rho^2}{1+\rho}\right] G_2(x)\, C(x) \tag{5.97}$$

Substituting (5.95) and (5.96) into (5.92), we obtain for $\hat{B}(x)$

$$\hat{B}(x) = \frac{\sigma_1^4}{\left[1 + 2\ (M-4)\epsilon + \left\{(M-4)\ (M-3) + 1\right\}\epsilon^2\right]} \left\{G_4(x)\, F_4(x)\right.$$
$$\left. + \frac{\epsilon G_2(x)\, F_2(x)}{\sigma_1^2} + \frac{[2\epsilon + (M-4)\epsilon^2]}{\sigma_1^4} \left[1 - \frac{2\rho^2}{1+\rho}\right] C(x)\, G_2(x)\right\} \tag{5.98}$$

Finally, when (5.98) and (5.97) are substituted into (5.87) and we let $\rho \to \frac{-1}{M-1}$, we obtain the desired result. <u>QED</u>

With these three relationships the fact that the regular simplex is a local maximum in the tangent plane defined by (5. 9) can now be proven.

To do this, define

$$b_i = \sum_{j=1}^{M} \left(\lambda'_{ij} - \lambda^R_{ij} \right) \tag{5. 99}$$

and note that

$$\frac{\partial \Phi(x;\alpha_R)}{\partial \lambda_{ij}} = G_2(x)\ F_2(x),\ i \neq j \tag{5. 100}$$

Then

$$a_1(x) = \tfrac{1}{2}\ G_2(x)\ F_2(x) \left(\sum_{i=1}^{M} b_i \right) \tag{5. 101}$$

Now we prove

Lemma 5. 8. For any admissible α'

$$a_2(x) = \left(\sum_{i>} \sum_{j} \gamma^2_{ij} \right) (p - 2q + r) + \tfrac{1}{4} \left(\sum_{i=1}^{M} b_i \right)^2 r + \left(\sum_{i=1}^{M} b_i^2 \right) (q - r) \tag{5. 102}$$

where

$$\gamma_{ij} = \lambda'_{ij} - \lambda^R_{ij} \tag{5. 103}$$

Proof. From (5.57)

$$a_2(x) = \sum_{i>} \sum_{j} \sum_{k>} \sum_{l} \gamma_{ij}\ \gamma_{kl}\ a_{ijkl}$$

which can be rewritten as

$$a_2(x) = \left(\sum_{i>j}\sum \gamma_{ij}^2\right) p + \sum_{i>j}\sum \sum_{\substack{k>l \\ (i,j)\neq(k,l)}}\sum \gamma_{ij}\,\gamma_{kl}\,a_{ijkl}$$

$$= \left(\sum_{i>j}\sum \gamma_{ij}^2\right) p + \frac{1}{4}\sum_{i>j}\sum \sum_{\substack{k>l \\ (i,j)\neq(k,l) \\ (i,j)\neq(l,k)}}\sum \gamma_{ij}\,\gamma_{kl}\,a_{ijkl}$$

$$= \left(\sum_{i>j}\sum \gamma_{ij}^2\right) p +$$

$$+ \sum_i \sum_j \gamma_{ij}\left[\frac{q}{2}\left(\sum_{\substack{k \\ k\neq j}} \gamma_{ik} + \sum_{\substack{k \\ k\neq i}} \gamma_{jk}\right) + \left(\sum_{\substack{k \\ k\neq i \\ k\neq j}}\sum_{\substack{l \\ l\neq i \\ l\neq j}} \gamma_{kl}\right)\frac{r}{4}\right] \tag{5.104}$$

The coefficient of q is

$$\frac{1}{2}\sum_i\sum_j \gamma_{ij}\left[\left(b_i - \gamma_{ij}\right) + \left(b_i - \gamma_{ij}\right)\right] = -2\sum_{i>j}\sum \gamma_{ij}^2 + \sum_i b_i^2$$

The coefficient of r is

$$\frac{1}{4}\sum_i\sum_j \gamma_{ij}\sum_{\substack{k \\ k\neq i \\ k\neq l}}\sum_{\substack{l \\ l\neq i \\ l\neq j}} \gamma_{kl} = \frac{1}{4}\sum_i\sum_j \gamma_{ij}\sum_{\substack{l \\ l\neq i \\ l\neq j}}\left[b_l - \gamma_{il} - \gamma_{jl}\right]$$

$$= \sum_{i>j}\sum \gamma_{ij}^2 + \frac{1}{4}\left(\sum_i b_i\right)^2 - \sum_i b_i^2$$

Substituting these into (5.104) gives the desired result. <u>QED</u>

Now restrict α' to lie in the tangent plane given by (5.9).

Therefore

$$\sum_i\sum_j \lambda'_{ij} = 0$$

which, employing Lemma 4.5, implies

$$\sum_{i} \lambda_{ij}^{'} = 0, \; j = 1, \ldots, M.$$

Hence

$$b_i = 0, \; i = 1, \ldots, M.$$

So, when α' is in this tangent plane, we have that

$$a_2(x) = \left[\sum_{i>} \sum_{j} \gamma_{ij}^{2} \right] (p - 2q + r) \tag{5.105}$$

Now

$$p - 2q + r = \left\{ \left[\left(\frac{M-1}{M-2} \right)^2 x^2 - \frac{2(M-1)}{M(M-2)} \right] F_2(x) + \frac{(M-1)}{M(M-2)} \; C(x) \right\} G_2(x)$$
$$+ \frac{2(M-1)}{(M-2)(M-3)} \; x \, G_3(x) \, F_3(x). \tag{5.106}$$

From (4.43) we can write

$$\phi(\lambda;\alpha') = \int_{0}^{\infty} e^{\lambda x} \frac{d}{dx} \left[\Phi(x;\alpha_R) \right] dx + \int_{-\infty}^{\infty} e^{\lambda x} \frac{d}{dx} \left[\Phi(x;\alpha') - \Phi(x;\alpha_R) \right] dx$$

which, using (4.44), can be expressed as

$$\phi(\lambda;\alpha') = \phi(\lambda;\alpha_R) - \lambda \int_{-\infty}^{\infty} e^{\lambda x} \left[\Phi(x;\alpha') - \Phi(x;\alpha_R) \right] \tag{5.107}$$

so that, employing (5.55), we have that

$$\phi(\lambda;\alpha') = \phi(\lambda;\alpha_R) - \lambda \int_{0}^{\infty} e^{\lambda x} \left\{ \sum_{k=1}^{\infty} a_k(x) \frac{\theta^k}{K!} \right\} dx \tag{5.108}$$

But $a_1(x)$ vanishes when α' is in the tangent plane. Hence, in order to verify that α_R is a local maximum in this tangent plane, we have only to prove the following.

Theorem 5.5.

$$\int_0^{\infty} e^{\lambda x} \, a_2(x) \, dx > 0 \qquad (5.109)$$

when α' is in the tangent plane defined by

$$\sum_i \sum_j \lambda_{ij} = 0$$

Proof. Since in (5.93)

$$\sum_{i>j} \sum \gamma_{ij}^2 > 0$$

the proof reduces to showing that

$$\int_0^{\infty} e^{\lambda x} \, (p - 2q + r) \, dx > 0 \qquad (5.110)$$

where (p - 2q +r) is given in (5.106). The integral over the last two terms of (p - 2q + r) is clearly positive. Hence it is sufficient to verify that

$$\int_0^{\infty} e^{\lambda x} \left[\left(\frac{M-1}{M-2} \right) x^2 - \frac{2}{M} \right] G_2(x) \, F_2(x) \, dx > 0 \qquad (5.111)$$

To do this, use the fact that

$$\frac{d}{dx} G_2(x) = -2 \left(\frac{M-1}{M-2} \right) x G_2(x)$$

to integrate

$$\int_0^{\infty} e^{\lambda x} \, G_2(x) \, F_2(x) \, dx$$

by parts, yielding the relationship

$$\left(\frac{M-1}{M-2} \right) \int_0^{\infty} e^{\lambda x} \, x^2 \, G_2(x) \, F_2(x) \, dx = \tfrac{1}{2} \int_0^{\infty} e^{\lambda x} \, G_2(x) \, F_2(x) \left[1 + x\lambda \right] dx$$
$$+ \tfrac{1}{2} \int_0^{\infty} x e^{\lambda x} \, G_2(x) \left[\frac{d F_2(x)}{dx} \right] dx \qquad (5.112)$$

which when substituted into (5.111), results in the inequality

$$\left[\frac{1}{2}-\frac{2}{M}\right]\int_0^\infty e^{\lambda x}\, G_2(x)\, F_2(x)\, dx + \int_0^\infty x e^{\lambda x} G_2(x)\left[\lambda F_2(x)+\frac{dF_2(x)}{dx}\right]dx > 0 \tag{5.113}$$

Both of these integrals are positive for $\lambda > 0$, and the coefficient of the first is positive for $M > 3$. Since the optimal solution for the $M = 3$ case has already been found, the proof is complete. QED

This completes the discussion of second order conditions for the regular simplex, from which we can conclude that for any α' in the tangent plane and in the neighborhood of α_R

$$\phi(\lambda;\alpha') < \phi(\lambda;\alpha_R)$$

In summary we have proven the following main results about the regular simplex signal structure.

(1) In Γ, the regular simplex is a local maximum at every signal-to-noise ratio and is the only signal structure that is a local maximum at all signal-to-noise ratio.

(2) The regular simplex is the global optimum for sufficiently small signal-to-noise ratio and for signal-to-noise ratio sufficiently large.

5.5 Maximizing the Minimum Distance

Because of its relative simplicity, the criterion of maximizing the minimum distance between the signal vectors has been a common one. In the class of all admissible signal sets we have the following:

Lemma 5.9. In the class, Γ, of all admissible α, the regular simplex is the only polytope which maximizes the minimum distance between the signal vectors.

Remark: In this case of no dimensionality restriction and for the case when D is restricted to two, the problem of maximizing the minimum distance has a unique solution. However, this criterion does not in general have a unique solution. In addition, different signal sets with the same minimum distance will be shown in the following chapter to have different probabilities of detection, thus making it a somewhat questionable criterion. All of the signal design solutions known to date, however, do maximize the minimum distance in the subclass of Γ in which they are the optimum.

In prior work maximizing the minimum distance had been the accepted criterion, so its relation with probability of detection ought to be known. It was most likely the first criterion used in signal design and appeared attractive because of its intuitive connection with maximum likelihood decision rules and the divergence criterion.

Proof:

Maximizing the minimum distance is synonymous to minimizing

$$\max_{i \neq j} \lambda_{ij} \tag{5.114}$$

The minimum value (5.114) we can attain is $\frac{-1}{M-1}$ since for any α

$$\sum_{i > j} \sum \lambda_{ij} \geq -\frac{M}{2} ,$$

Thus, we must show α_R is the only polytope which attains this minimum value. Assume α represents another polytope with

$$\max_{i \neq j} \lambda_{ij} = -\frac{1}{M-1}$$

and reorder the λ_{ij} so that

$$\lambda_{12} = \max_{i \neq j} \lambda_{ij}$$

We know that $D(\alpha) = 0$, since it has already been shown that

$$\sum_{i>j}\sum \lambda_{ij} > -\frac{M}{2}$$

whenever $D(\alpha) > 0$ which implies

$$\max_{i \neq j} \lambda_{ij} > \frac{-1}{M-1}$$

Therefore, for α

$$\sum \sum \lambda_{ij} = -\frac{M}{2}$$

from which

$$\sum_{i=1}^{M} \lambda_{1i} = 0$$

or

$$\lambda_{13} + \ldots + \lambda_{1M} = -\left(\frac{M-2}{M-1}\right)$$

If any

$$\lambda_{1i} \quad i \geq 3$$

is less than

$$\frac{-1}{M-1}$$

then at least one of the others has to be greater than this term, which is a contradiction.

Then

$$\lambda_{1i} = \frac{-1}{M-1} \quad i \geq 2.$$

Using this and repeating the argument for

$$\sum_{j=1}^{M} \lambda_{ij} = 0 \quad i \geq 2$$

we see that

$$\lambda_{ij} = \frac{-1}{M-1} \quad \text{for all } i \neq j \qquad \underline{\text{QED}}$$

REFERENCES

5.1 Balakrishnan, A. V., "A Contribution to the Sphere Packing Problem of Communication Theory", Journal of Math Analysis and Applications, Vol. 3, pp. 485-506, December, 1961.

5.2 Ziv, J., "Generation of Optimal Codes by a 'Pyramid-Packing' Argument", IEEE Trans. on Information Theory, Vol. II-10, No. 3, pp. 253-255, July, 1964.

5.3 Balakrishnan, A. V., "Advances in Communication", Academic Press, 1965.

5.4 Stutt, C. A., "Regular Polyhedron Codes", Research Report No. 59-R6-2202, General Electric Research Laboratory, March, 1959.

5.5 Balakrishnan, A. V., and J. E. Taber, "Error Rates in Coherent Communication Systems", IRE Trans. on Communication Systems, CS-10: 86-90, March, 1962.

5.6 Nuttall, A. H., "Error Probabilities for Equicorrelated M-ary Signals Under Phase Coherent and Phase Incoherent Reception", IRE Trans. on Information Theory, IT-8: 305-315, July, 1962.

5.7 Max, J., "Signals Sets With Uniform Correlation Properties", J. SIAM, 10: 113-118, March, 1962.

5.8 Stutt, C. A., "Information Rate in a Continuous Channel for Regular Simplex Codes", IRE Trans. on Information Theory, IT-6: 516-522, December, 1960.

5.9 Chang, S. S. L., B. Harris, and J. J. Metzner, "Optimum Message Transmission in a Finite Time", IRE Trans. on Information Theory, IT-8: 215-224, September, 1962.

5.10 Golomb, S. W., Baumert, L. D., Easterling, M. F., Stiffler, J. J., and Viterbi, A. J., "Digital Communications", Prentice Hall, 1964.

VI. OPTIMALITY WHEN THE DIMENSIONALITY IS RESTRICTED TO $D \leq M-2$

The results derived in the previous chapter indicate that when dimensionality is unrestricted, the optimal signal set is the regular simplex which requires M-1 degrees of freedom.

We now consider the problem of determining optimal signal sets when the allowed degrees of freedom are restricted to be less than M-1. In particular, we now reduce the allowed degrees of freedom from that required for the regular simplex by one, and look for optimal signal sets when $D \leq M-2$, which corresponds to an equivalent reduction on the allowed bandwidth.

In this case (i.e., in the class of admissible α for which $D \leq M-2$), the signal sets which have inner product matrices of the form

$$\alpha_o = \left(\begin{array}{c|c} \begin{matrix} 1 & & \frac{-1}{M_1-1} \\ & \ddots & \\ \frac{-1}{M_1-1} & & 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 1 & & \frac{-1}{M_2-1} \\ & \ddots & \\ \frac{-1}{M_2-1} & & 1 \end{matrix} \end{array} \right) \tag{6.1}$$

where M_1 can assume the values $2, 3, \ldots, M-2$, and $M_1+M_2=M$, are local extrema for all signal-to-noise ratios; that is, they satisfy necessary (first order) conditions to be optimal. This is demonstrated in Section 6.1.

These local extrema consist of two regular simplices placed orthogonal to one another, with dimensionalities M_1-1 and M_2-1, thus satisfying the restriction of M signals in M-2 (or less) dimensions. These necessary conditions provide M-3 local extrema which must be classified into local maxima, local minima, or saddle points.

However, in Section 6.2, we show that each of the above extrema satisfies sufficient (second order) conditions to be a local maximum in Γ_2, where for convenience we define Γ_2 as that subclass of Γ for which $D \leq M-2$. Finally, it is shown that the largest of the M-3 local maxima satisfies the condition:

$$\left|M_1 - M_2\right| - 1. \tag{6.2}$$

For each M, this condition is satisfied by only one of the local maxima. Thus, for M even, the largest local maximum consists of two regular simplices, each formed from $\frac{M}{2}$ signals; and for M odd, one is formed from $\frac{M-1}{2}$ signals and one from $\frac{M+1}{2}$.

It should not be surprising that optimal signal sets are not equi-correlated for the cases when the dimensionality restrictions are less than M-1, since the only signal set which is equi-correlated and does not have dimensionality D = M is α_R, for which D = M-1. Hence, these optimal sets necessarily cannot be equi-correlated.

As a special case of interest, these results show that for 5 points in 3 dimensions, the choice which corresponds to placing 4 points on the equator and one on the pole is not optimum, while the choice of 3 points on the equator and one on each pole is the optimum, even though both have the same minimum distance, thus settling another heretofore unresolved point in this theory. In particular, we note that the mean width is larger in the latter case.

6.1 Necessary (First Order) Conditions

We are looking for admissible α which maximize the probability of detection subject to the restriction that $D \leq M-2$. In this section necessary conditions which a local extremum must satisfy are attained, and those α_o in Equation (6.1) are shown to satisfy these conditions.

First, note that from (4.44) the probability of detection can be written

$$P_D(\lambda;\alpha) = e^{-\lambda^2/2} \int_o^\infty e^{\lambda x} \; G(x) \left[\phi(x) \right]^{M-1} dx$$

$$- \frac{\lambda}{M} e^{-\lambda^2/2} \int_o^\infty e^{\lambda x} \left\{ \Phi(x, \alpha) - \left[\phi(x) \right]^M \right\} dx \tag{6.3}$$

where, as before, we integrate only over the region (o, ∞) since we already know that the optimal solutions have convex hulls which contain the origin. Hence minimizing

$$J\ (\lambda;\alpha) = \int_o^\infty e^{\lambda x} \left\{ \Phi(x;\alpha) - \left[\phi(x) \right]^M \right\} dx \tag{6.4}$$

is equivalent to maximizing $P_D(\lambda;\alpha)$.

Second, note that the M^2 constraint equations

$$C_{ij} = 0 \qquad i, j = 1, \ldots, M, \tag{6.5}$$

where as before, C_{ij} is the cofactor (including sign) of λ_{ij} in α, are sufficient to restrict α to M-2 degrees of freedom. Since the α space consists only of symmetric matrices, the number of constraint equations can be reduced to

$$C_{ij} = 0 \quad \text{for} \quad i \geq j \quad j = 1, \ldots, M \tag{6.6}$$

with no loss of generality.

Let Δ_2 denote the class of all α whose elements satisfy (6.6). Observe that Δ_2 includes non-admissible as well as admissible α; that is, there may be α satisfying (6.6) that are not non-negative definite.

But

$$\Gamma_2 = \Delta_2 \cap \Gamma \tag{6.7}$$

In addition, if extrema are found in Δ_2, and they are admissible, i.e., they are also elements of Γ_2, then they are also extrema of Γ_2. This is the approach we adopt.

Now, consider some α' in Γ_2, i.e., α' is a point on each of the surfaces in α-space defined by $C_{ij} = 0$; $i \geq j$, $j = 1, \ldots, M$. The vector normal to the surface $C_{ij} = 0$ at α' is given by the $\left(\frac{M(M-1)}{2}\right) \times 1$ gradient vector:

$$\nabla C_{ij}\Big|_{\alpha'} = \begin{pmatrix} \left.\dfrac{\partial C_{ij}}{\partial \lambda_{12}}\right|_{\alpha'} \\ \left.\dfrac{\partial C_{ij}}{\partial \lambda_{13}}\right|_{\alpha'} \\ \cdot \\ \cdot \\ \cdot \\ \left.\dfrac{\partial C_{ij}}{\partial \lambda_{M-1,M}}\right|_{\alpha'} \end{pmatrix} \tag{6.8}$$

and the plane tangent to the surface at α' consists of those vectors t satisfying

$$\left(\nabla C_{ij}\Big|_{\alpha'} \cdot t \right) = 0 \tag{6.9}$$

If α' is to be a local minimum with respect to points in the neighborhood of α' satisfying $C_{ij} = 0$, for all (i, j), then

$$\left(\nabla \phi(\lambda;\alpha') \cdot t \right) = 0 \tag{6.10}$$

for all t satisfying (6.9), or equivalently

$$\left(\nabla J(\lambda;\alpha') \cdot t \right) = 0 \tag{6.11}$$

where

$$\nabla J(\lambda;\alpha') = \begin{pmatrix} \dfrac{\partial J(\lambda;\alpha')}{\partial\lambda_{12}} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial J(\lambda;\alpha')}{\partial\lambda_{M-1,M}} \end{pmatrix} = \frac{-1}{\lambda} \begin{pmatrix} \dfrac{\partial\phi(\lambda;\alpha')}{\partial\lambda_{12}} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial\phi(\lambda;\alpha')}{\partial\lambda_{M-1,M}} \end{pmatrix} = \frac{-1}{\lambda}\nabla\phi(\lambda;\alpha') \tag{6.12}$$

and where $\phi(\lambda;\alpha)$ is defined by (4.12).

For if (6.11) is not satisfied, there would then exist a curve in $C_{ij} = 0$ through α' on which the projection of $\nabla J(\lambda;\alpha')$ would be negative, thus decreasing $J(\lambda;\alpha)$. The existence of such a curve is guaranteed since C_{ij} is a polynomial in the $\lambda_{k\ell}$, and therefore differentiable. We have, however, $\frac{M(M+1)}{2}$ surfaces and corresponding $\frac{M(M+1)}{2}$ tangent planes to which $\nabla J(\lambda;\alpha')$ must be orthogonal, if α' is to be a local minimum of $J(\lambda;\alpha)$ in Γ_2. The flat which is tangent to all the surfaces of α' consists of those vectors t such that

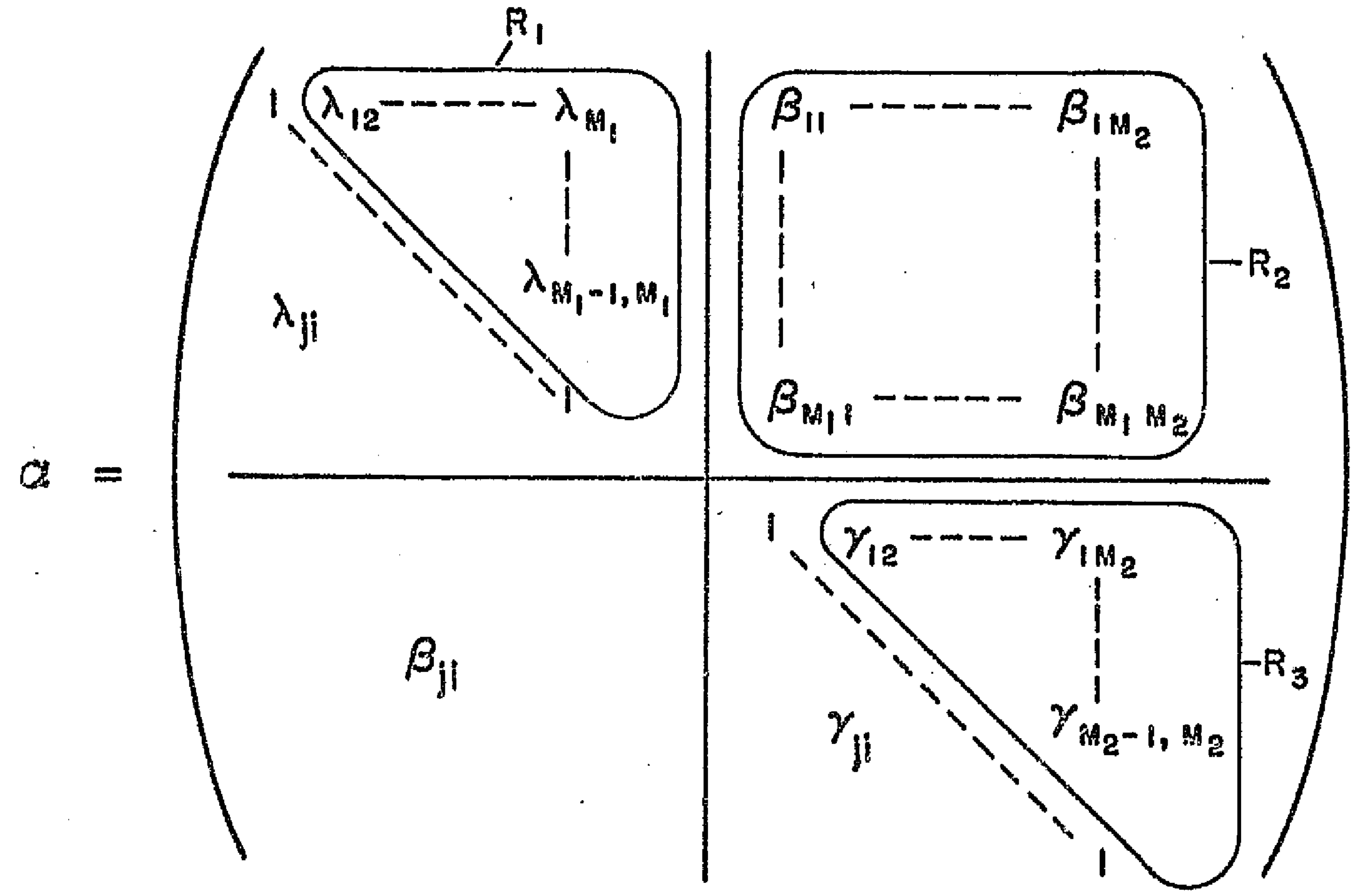
$$\left(\nabla C_{ij}\Big|_{\alpha'} \cdot t\right) = 0 \quad \text{for} \quad i \geq j, \quad j = 1, \ldots, M \tag{6.13}$$

Note that as the dimensionality of the manifold spanned by the set of normal vectors $\left\{\nabla C_{ij}\right\}$ increases, the dimensionality of the flat defined by (6.13) decreases. If α' is to be a local minimum with respect to those α which are in all $\frac{M(M+1)}{2}$ surfaces, then the relationship

$$\left(\nabla J(\lambda;\alpha') \cdot t\right) = 0 \tag{6.14}$$

must hold for all t in the flat defined in (6.13). Finally, since the flat defined in (6.13) is orthogonal to the manifold spanned by the normal vectors $\left\{\nabla C_{ij}\right\}$, and their union is the whole α-space, we can conclude (6.14) will be satisfied if, and only if, $\nabla J(\lambda;\alpha')$ can be expressed as a linear combination of the ∇C_{ij}, that is, if

$$\nabla J(\lambda;\alpha') = \sum_{ij}^{M} \nu_{ij} \nabla C_{ij}\Big|_{\alpha'} \tag{6.15}$$



IDENTIFICATION AND INDEXING DIAGRAM FOR AN ARBITRARY α AND FIXED M_1 AND M_2

FIGURE 6.1

Equation (6.15) is the necessary condition that must be satisfied by α' for it to be a local extremum in Γ_2.

<u>Theorem 6.1.</u>

The α_o given by

$$\alpha_o = \left(\begin{array}{ccc|ccc} 1 & & \frac{-1}{M_1-1} & & & \\ & \ddots & & & 0 & \\ \frac{-1}{M_1-1} & & 1 & & & \\ \hline & & & 1 & & \frac{-1}{M_2-1} \\ & 0 & & & \ddots & \\ & & & \frac{-1}{M_2-1} & & 1 \end{array}\right) \qquad (6.1)$$

where $M_1 = 2, \ldots, M-2$,

and $M_1 + M_2 = M$

satisfy the necessary condition in (6.15) and are therefore local extrema in Γ_2.

Proof:

It is immediate that these α_o are admissible and have dimensionality equal to M-2.

Henceforth, for M_1 and M_2 fixed such that $M_1 + M_2 = M$, we will identify and index the elements of an arbitrary α as indicated in Figure 6.1. Identify the λ_{ij} elements as being in R_1, the β_{ij} in R_2, and the γ_{ij} in R_3. Assume components of the gradient and normal vectors are arranged in the following order: First over R_1, then R_2, and finally over R_3.

Denote the $M_1 \times M_1$ regular simplex by

$$\alpha_{R_{M_1}} = \begin{pmatrix} 1 & & \frac{-1}{M_1-1} \\ & \ddots & \\ \frac{-1}{M_1-1} & & 1 \end{pmatrix} \qquad (6.16)$$

and similarly the $M_2 \times M_2$ regular simplex by

$$\alpha_{R_{M_2}} = \begin{pmatrix} 1 & & \frac{-1}{M_2-1} \\ & \ddots & \\ \frac{-1}{M_2-1} & & 1 \end{pmatrix} \qquad (6.17)$$

Hence

$$\alpha_o = \left(\begin{array}{c|c} \alpha_{R_{M_1}} & 0 \\ \hline 0 & \alpha_{R_{M_2}} \end{array}\right) \tag{6.18}$$

where $M_1 + M_2 = M$, and using (4.49)

$$J(\lambda;\alpha_o) = \int_0^\infty e^{\lambda x} \Bigg\{ \underbrace{\int_{-\infty}^{x} \cdots \int}_{M\text{-fold}} G(\xi;0;\alpha_o)\, d\,|\xi| - \left[\phi(x)\right]^M \Bigg\}\, dx$$

$$= \int_0^\infty e^{\lambda x} \left[\Bigg\{ \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_1\text{-fold}} G\left(\xi;0;\alpha_{R_{M_1}}\right) d|\xi| \Bigg\} \cdot \right.$$

$$\left. \cdot \Bigg\{ \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_2\text{-fold}} G\left(\eta;0;\alpha_{R_{M_2}}\right) d|\eta| \Bigg\} - \{\phi(x)\}^M \right] dx \tag{6.19}$$

Because of the symmetry in $\alpha_{R_{M_1}}$ and $\alpha_{R_{M_2}}$ (and introducing the symbols c_1, c_2, and c_3), we can write

$$\frac{\partial J(\lambda;\alpha_o)}{\partial \lambda_{ij}} = \frac{\partial J(\lambda;\alpha_o)}{\partial \lambda_{12}} = c_1 = \int_0^\infty e^{\lambda x}\, dx.$$

$$\Bigg\{ \underbrace{\int_{-\infty}^{x} \cdots \int}_{(M_1-2)\text{-fold}} G\left(x, x, \xi_3, \xi_4, \ldots, \xi_{M_1}; 0; \alpha_{R_{M_1}}\right) d\xi_3 \ldots d\xi_{M_1} \Bigg\} \cdot$$

$$\Bigg\{ \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_2\text{-fold}} G\left(\eta; 0; \alpha_{R_{M_2}}\right) d|\eta| \Bigg\} \tag{6.20}$$

for all λ_{ij} in R_1. Also,

$$\frac{\partial J(\lambda;\alpha_o)}{\partial \gamma_{ij}} = \frac{\partial J(\lambda;\alpha_o)}{\partial \gamma_{12}} = c_3 = \int_o^\infty e^{\lambda x}\, dx$$

$$\cdot \left\{ \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_1\text{-fold}} G\left(\xi;\ 0;\ \alpha_{R_{M_1}}\right) d\,|\xi| \right\}.$$

$$\cdot \left\{ \underbrace{\int_{-\infty}^{x} \cdots \int}_{(M_2-2)\text{-fold}} G\left(x, x, \eta_3, \ \eta_4, \ldots, \eta_{M_2};\ 0;\ \alpha_{R_{M_2}}\right) d\eta_3 \ldots d\eta_{M_2} \right\} \tag{6.21}$$

for all γ_{ij} in R_3; and

$$\frac{\partial J(\lambda;\alpha_o)}{\partial \beta_{ij}} = \frac{\partial J(\lambda;\alpha_o)}{\partial \beta_{11}} = c_2 = \int_o^\infty e^{\lambda x}\, dx$$

$$\cdot \left\{ \underbrace{\int_{-\infty}^{x} \cdots \int}_{(M_1-1)\text{-fold}} G\left(x, \xi_2, \ldots, \xi_{M_1};\ 0;\alpha_{R_{M_1}}\right) d\,\xi_2 \ldots d\xi_{M_1} \right\}$$

$$\cdot \left\{ \underbrace{\int_{-\infty}^{x} \cdots \int}_{(M_2-1)\text{-fold}} G\left(x, \eta_2, \ldots, \eta_{M_2};\ 0;\alpha_{R_{M_2}}\right) d\eta_2 \ldots d\eta_{M_2} \right\} \tag{6.22}$$

for all β_{ij} in R_2. Therefore, the gradient vector at α_o is of the form

$$\nabla J(\lambda;\alpha_o) = \begin{bmatrix} \dfrac{\partial J(\lambda;\alpha_o)}{\partial \lambda_{12}} \\ \vdots \\ \dfrac{\partial J(\lambda;\alpha_o)}{\partial \lambda_{M_1-1,\, M_1}} \\ \dfrac{\partial J(\lambda;\alpha_o)}{\partial \beta_{11}} \\ \vdots \\ \dfrac{\partial J(\lambda;\alpha_o)}{\partial \beta_{M_1 M_2}} \\ \dfrac{\partial J(\lambda;\alpha_o)}{\partial \gamma_{12}} \\ \vdots \\ \dfrac{\partial J(\lambda;\alpha_o)}{\partial \gamma_{M_2-1,\, M_2}} \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_1 \\ c_2 \\ \vdots \\ c_2 \\ c_3 \\ \vdots \\ c_3 \end{bmatrix} \tag{6.23}$$

We now look at the form of the $\left\{ \nabla C_{ij} \right\}$ at α_o.

Define $C_{ij}^{k\ell}$ to be the determinant of α after removing the i^{th} and k^{th} rows, the j^{th} and ℓ^{th} columns, and multiplying by $(-1)^{i+j+k+\ell}$.

Also define $C_{ij}^{k\ell mn}$ to be the determinant of α after removing the i^{th}, k^{th}, and m^{th} rows, the j^{th}, ℓ^{th}, and n^{th} columns, and multiplying by $(-1)^{i+j+k+\ell+m+n}$.

Employing Lemma 4.6, the following relationships for any α can be readily verified:

$$\frac{\partial C_{ij}}{\partial \lambda_{ij}} = C_{ii}^{jj}, \qquad i \neq j \tag{6.24}$$

$$\frac{\partial C_{jj}}{\partial \lambda_{ij}} = 0, \qquad i \neq j \tag{6.25}$$

$$\frac{\partial C_{ij}}{\partial \lambda_{jk}} = C_{ij}^{jk} = C_{jj}^{ik}, \; i \neq j, \; j \neq k, \; k \neq i \tag{6.26}$$

$$\frac{\partial C_{jj}}{\partial \lambda_{k\ell}} = 2\, C_{jj}^{k\ell}, \; k \neq \ell, \; \ell \neq j, \; j \neq k \tag{6.27}$$

and

$$\frac{\partial C_{ij}}{\partial \lambda_{k\ell}} = \sum_{\substack{m=1 \\ m \neq i \\ m \neq k}}^{M} C_{ij}^{\substack{m\ell \\ kk}} \text{ all indices different.} \tag{6.28}$$

We will use the fact that the N x N matrix

$$\alpha_\rho = \begin{pmatrix} 1 & & \rho \\ & \ddots & \\ \rho & & 1 \end{pmatrix} \tag{6.29}$$

has diagonal cofactors

$$C_{ii}(\rho) = \Big[1 + (N-2)\, \rho \Big] (1-\rho)^{N-2}_{\; i = 1, \ldots, N} \tag{6.30}$$

and off diagonal cofactors

$$C_{ij}(\rho) = -\, \rho (1-\rho)^{N-2} \quad i \neq j \tag{6.31}$$

Applying (6.24) to (6.31) to α_o, the following relationships can be readily verified, to which we assign the symbols b_o, $b_1, \ldots, b_6$:

$$C_{ii}^{jj}\Big|_{\alpha_o} = \begin{cases} 0 & \text{if } i \text{ and } j \le M_1 \\ & \text{or } i \text{ and } j > M_1 \\ C_{ii_{M_1}} \cdot C_{jj_{M_2}} = b_o \neq 0 & \\ \quad \text{if } i \le M_1, \text{ and } j > M_1 & \end{cases} \tag{6.32}$$

where $C_{ii_{M_1}}$ is the diagonal cofactor of $\alpha_{R_{M_1}}$, and $C_{ii_{M_2}}$ the diagonal cofactor of $\alpha_{R_{M_2}}$.

$$C_{jj}^{k\ell} = \begin{cases} C_{jj_{M_1}} \cdot C_{k\ell_{M_2}} = b_1 & \text{if } jj \text{ is in } R_1 \text{ and } k\ell \text{ in } R_2 \\ C_{jj_{M_2}} \cdot C_{k\ell_{M_1}} = b_2 & \text{if } jj \text{ is in } R_2 \text{ and } k\ell \text{ in } R_1 \\ 0 & \text{if } jj \text{ is in } R_1 \text{ or } R_3 \text{ and } k\ell \text{ in } R_2 \end{cases} \tag{6.33}$$

if all indices are $\le M_1$ or $> M_1$, i.e., if all indices are in R_1 or all indices are in R_2.

$$\frac{\partial C_{ij}}{\partial \lambda_{k\ell}} = \begin{cases} \left(C_{ij_{M_2}}\right)(M_1)\left(C^{m\ell}_{kk_{M_1}}\right) = b_3 \neq 0 & \text{if } k\ell \text{ is in } R_1 \text{ and } ij \text{ is in } R_3 \\ \left(C_{ij_{M_1}}\right)(M_2)\left(C^{m\ell}_{kk_{M_2}}\right) = b_4 \neq 0 & \text{if } k\ell \text{ is in } R_3 \text{ and } ij \text{ is in } R_1 \\ \left(C^{i\ell}_{kk_{M_1}}\right)\left[C_{jj_{M_2}} + (M_2-1)C_{mj_{M_2}}\right] = & b_5 \text{ if } k\ell \text{ is in } R_1 \text{ and } ij \text{ is in } R_2, \\ & \text{or if } k\ell \text{ is in } R_2 \text{ and } ij \text{ is in } R_1 \\ \left(C^{i\ell}_{kk_{M_2}}\right)\left[C_{jj_{M_1}} + (M_1-1)C_{mj_{M_1}}\right] = & b_6 \neq 0 \text{ if } k\ell \text{ is in } R_3 \text{ and } ij \text{ is in} \\ & R_2, \text{ or if } k\ell \text{ is in } R_2 \text{ and } ij \text{ is in } R_3 \end{cases} \tag{6.34}$$

In Figure 6. 2, enough of the normal vectors are described to form $\nabla J(\lambda;\alpha_o)$. The R_1 portion of $\nabla J(\lambda;\alpha_o)$ comes from ∇C_{ii} where ii is in R_3; and the R_3 part of $\nabla J(\lambda;\alpha_o)$ can come from ∇C_{ii}, for ii in R_1. By summing ∇C_{ij} (ij in R_1) over one row in R_1, there results

	$\nabla\phi$	∇C_{ii} ii in R_1	∇C_{ii} ii in R_3	∇C_{ij} ij in R_2	∇C_{ij} ij in R_1
R_1	c_1 ⋮ c_1	0 ⋮ 0	$2b_2$ ⋮ $2b_2$	b_5 ⋮ b_5	0 ⋮ 0
R_2	c_2 ⋮ c_2	0 ⋮ 0	0 ⋮ 0	0 ⋮ 0	b_5 ⋮ b_5 0 ⋮ 0 } ith row b_5 ⋮ b_5
R_3	c_3 ⋮ c_3	$2b_1$ ⋮ $2b_1$	0 ⋮ 0	b_6 ⋮ b_6	b_6 ⋮ b_6

NORMAL VECTORS NEEDED TO FORM $\nabla J(\lambda;\alpha_o)$

FIGURE 6. 2

$$\sum_{i=1}^{M_1} \nabla C_{ij}\Big|_{ij \text{ in } R_1} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline (M_1-1)\, b_5 \\ \cdot \\ \cdot \\ \cdot \\ (M_1-1)\, b_5 \\ \hline M_1\, b_6 \\ \cdot \\ \cdot \\ \cdot \\ M_1\, b_6 \end{pmatrix} \qquad (6.35)$$

By subtracting from this a scalar times ∇C_{ii}(ii in R_2), we have a vector of the form

$$\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline b \\ \cdot \\ \cdot \\ \cdot \\ b \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \qquad (6.36)$$

from which the R_2 part of $\nabla J(\lambda; \alpha_o)$ can be written.

This shows that $\nabla J(\lambda;\alpha_o)$ can be expressed as a linear combination of the normal vectors at α_o, and therefore proves that these α_o do satisfy the necessary first order conditions to be local minima of $J(\lambda;\alpha)$ in Γ_2. QED

To this point we have established that these α_o are local extrema in the class of admissible α for which $D \le M - 2$. Next, sufficient (that is, second order) conditions must be considered in order to conclude that these local extrema are indeed local minimum of $J(\lambda;\alpha)$.

6.2 Sufficient (Second Order) Conditions

The results obtained from the second order variations about the regular simplex are now used to determine second order variations for the α_o of the previous section.

Note that α_o is in the plane defined by

$$\sum_{i=1}^{M}\sum_{j=1}^{M} \lambda_{ij} = 0 \tag{6.37}$$

and further that at α_o the only admissible directions are in this tangent plane and in the boundary of Γ, where by admissible we now mean directions or paths which restrict α to Γ_2. This is the case since directions towards the interior of Γ from α_o have dimensionality $D = M$.

Let α' be any other admissible signal set in the neighborhood of α_o for which we write

$$J(\lambda;\alpha') - J(\lambda;\alpha_o) = \sum_{R_1} (\lambda'_{ij} - \mathring{\lambda}_{ij})\, c_1 + \sum_{R_2} (\beta'_{ij} - \mathring{\beta}_{ij})\, c_2 + \sum_{R_3} (\gamma'_{ij} - \mathring{\gamma}_{ij})\, c_3 + \ldots \tag{6.38}$$

where c_1, c_2, and c_3 are given by (6.20), (6.21), and (6.22), and are non-negative.

For α_o, we have that

$$\sum_{R_1} \mathring{\lambda}_{ij} + \sum_{R_2} \mathring{\beta}_{ij} + \sum_{R_3} \mathring{\gamma}_{ij} = -\frac{M}{2} \tag{6.39}$$

$$\sum_{R_1} \lambda_{ij}^{\circ} = -\frac{M_1}{2}, \tag{6.40}$$

$$\sum_{R_2} \beta_{ij}^{\circ} = 0, \tag{6.41}$$

and

$$\sum_{R_3} \gamma_{ij}^{\circ} = -\frac{M_2}{2} \tag{6.42}$$

For α', using Theorem 5.1, we can say

$$\sum_{R_2} \lambda_{ij}' + \sum_{R_2} \beta_{ij}' + \sum_{R_3} \gamma_{ij}' \geq -\frac{M}{2} \tag{6.43}$$

$$\sum_{R_1} \lambda_{ij}' \geq -\frac{M_1}{2} \tag{6.44}$$

and

$$\sum_{R_3} \gamma_{ij}' \geq -\frac{M_2}{2} \tag{6.45}$$

If equality exists in (6.44) and (6.45), then

$$\sum_{R_2} \beta_{ij}' \geq 0 \tag{6.46}$$

from which we have that the first order variations in (6.38) are non-negative.

It is immediate that there are cases for which there is strict inequality in (6.46). Hence, α_o cannot be a local minimum of probability of detection in Γ_2 and is therefore a local maximum or possibly a saddle point.

In order to conclude that α_o is indeed a local maximum in Γ_2 (local minimum of $J(\lambda;\alpha_o)$), we examine the second order variations when the first order variations vanish.

It is clear that the first order variations in (6.38) vanish when there is equality in (6.44), (6.45) and (6.46), for which we have the following.

Theorem 6.2. The second order variations in the neighborhood of α_o of $P_D(\lambda;\alpha_o)$ are negative for all signal-to-noise ratio in all admissible directions in Γ_2 for which the first order variations vanish and

$$\sum_{R_1} \lambda'_{ij} = -\frac{M_1}{2}$$

$$\sum_{R_2} \beta'_{ij} = 0$$

$$\sum_{R_3} \gamma'_{ij} = -\frac{M_2}{2} \tag{6.47}$$

Proof:

As before, we look at the second order variations of $J(\lambda;\alpha_o)$ and prove that they are positive.

Employing Lemma 4.5, using (6.47), and noting that (6.47) implies equality in (6.43) we have that

$$\sum_{i} \beta'_{ij} = 0 \qquad j = 1, \ldots, M_2;$$

and

$$\sum_{j} \beta'_{ij} = 0 \qquad i = 1, \ldots, M_1 \tag{6.48}$$

We now make the following definitions for

$$\Phi(x;\alpha_o) = \underbrace{\int_{-\infty}^{x}\!\!\cdots\!\int}_{M_1\text{-fold}} G\left(\xi; 0; \alpha_{R_{M_1}}\right) d\,|\xi| \underbrace{\int_{-\infty}^{x}\!\!\cdots\!\int}_{M_2\text{-fold}} G\left(\eta; 0; \alpha_{R_{M_2}}\right) d|\eta|$$

$$= \Phi\left(x;\alpha_{R_{M_1}}\right) \Phi\left(x;\alpha_{R_{M_2}}\right) \tag{6.49}$$

Define

$$p_1 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \lambda_{ij}^2} = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \lambda_{12}^2}$$

$$= \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_1\text{-fold}} \frac{\partial^4}{\partial \xi_1^2 \, \partial \xi_2^2} \, G\left(\xi; 0; \alpha_{R_{M_1}}\right) d\,|\xi| \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_2\text{-fold}} G\left(\eta; 0; \alpha_{R_{M_2}}\right) d\,|\eta| \tag{6.50}$$

$$p_2 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \gamma_{ij}^2} = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \gamma_{12}^2}$$

$$= \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_1\text{-fold}} G\left(\xi; 0; \alpha_{R_{M_1}}\right) d\,|\xi| \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_2\text{-fold}} \frac{\partial^4}{\partial \eta_1^2 \, \partial \eta_2^2} \, G\left(\eta; 0; \alpha_{R_{M_2}}\right) d\,|\eta| \tag{6.51}$$

$$q_1 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \lambda_{12} \, \partial \lambda_{13}}, \quad r_1 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \lambda_{12} \, \partial \lambda_{34}} \tag{6.52}$$

$$q_2 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \gamma_{12} \, \partial \gamma_{13}}, \quad r_2 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \gamma_{12} \, \partial \gamma_{34}} \tag{6.53}$$

$$s_1 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \lambda_{ij} \, \partial \beta_{ii}} = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \lambda_{12} \, \partial \beta_{11}}$$

$$= \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_1\text{-fold}} \frac{\partial^3}{\partial \xi_1^2 \, \partial \xi_2} \, G\left(\xi; 0; \alpha_{R_{M_1}}\right) d\,|\xi| \underbrace{\int_{-\infty}^{x} \cdots \int}_{M_2-1\text{ fold}} G\left(x, \eta_2, \ldots, \eta_{M_2}; 0; \alpha_{R_{M_2}}\right)$$

$$d\eta_2 \ldots d\eta_M \tag{6.54}$$

$$s_2 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial \gamma_{12} \, \partial \beta_{11}} \tag{6.55}$$

$$t_1 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial\lambda_{12}\,\partial\beta_{31}} \;;\; t_2 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial\gamma_{12}\,\partial\beta_{13}} \tag{6.56}$$

$$u = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial\lambda_{12}\,\partial\gamma_{12}} \;;\; v = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial\beta_{ij}^2} \tag{6.57}$$

and finally

$$w_1 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial\beta_{11}\,\partial\beta_{21}} \;;\; w_2 = \frac{\partial^2 \Phi(x;\alpha_o)}{\partial\beta_{11}\,\partial\beta_{12}} \tag{6.58}$$

With a_{ijkl} defined as in (5.58) with α_R replaced by α_o, then the second order variation of

$J(\lambda;\alpha') - J(\lambda;\alpha_o)$ can be written as

$$\left[\sum_{R_1}\left(\lambda'_{ij} - \lambda^\circ_{ij}\right) + \sum_{R_2}\left(\beta'_{ij} - \beta^\circ_{ij}\right) + \sum_{R_3}\left(\gamma'_{ij} - \gamma^\circ_{ij}\right)\right]\left[\sum_{R_1}\left(\lambda'_{kl} - \lambda^\circ_{kl}\right) + \sum_{R_2}\left(\beta'_{kl} - \beta^\circ_{kl}\right)\right.$$
$$\left. + \sum_{R_3}\left(\gamma'_{kl} - \gamma^\circ_{kl}\right)\right] a_{ijkl} = S_1 + S_2 + S_3 + 2\left[S_4 + S_5 + S_6\right] \tag{6.59}$$

where

$$S_1 = \left[\sum_{R_1}\left(\lambda'_{ij} - \lambda^\circ_{ij}\right)\right]\left[\sum_{R_1}\left(\lambda'_{kl} - \lambda^\circ_{kl}\right)\right] a_{ijkl} \tag{6.60}$$

$$S_2 = \left[\sum_{R_2}\left(\beta'_{ij} - \beta^\circ_{ij}\right)\right]\left[\sum_{R_2}\left(\beta'_{kl} - \beta^\circ_{kl}\right)\right] a_{ijkl} \tag{6.61}$$

$$S_3 = \left[\sum_{R_3}\left(\gamma'_{ij} - \gamma^\circ_{ij}\right)\right]\left[\sum_{R_3}\left(\gamma'_{kl} - \gamma^\circ_{kl}\right)\right] a_{ijkl} \tag{6.62}$$

$$S_4 = \left[\sum_{R_1}\left(\lambda'_{ij} - \lambda^\circ_{ij}\right)\right]\left[\sum_{R_2}\left(\beta'_{kl} - \beta^\circ_{kl}\right)\right] a_{ijkl} \tag{6.63}$$

$$S_5 = \left[\sum_{R_1}\left(\lambda'_{ij} - \lambda^\circ_{ij}\right)\right]\left[\sum_{R_3}\left(\gamma'_{kl} - \gamma^\circ_{kl}\right)\right] a_{ijkl} \tag{6.64}$$

$$S_6 = \left[\sum_{R_2}\left(\beta'_{ij} - \beta^\circ_{ij}\right)\right]\left[\sum_{R_3}\left(\gamma'_{kl} - \gamma^\circ_{kl}\right)\right] a_{ijkl} \tag{6.65}$$

Now

$$S_5 = u \left[\sum_{R_1} \left(\lambda'_{ij} - \lambda^\circ_{ij} \right) \right] \left[\sum_{R_3} \left(\gamma'_{kl} - \gamma^\circ_{kl} \right) \right] \tag{6.66}$$

and applying (6.40), (6.42) and (6.47) it is immediate that $S_5 = 0$.

For S_4, we have

$$S_4 = \sum_{R_1} \left(\lambda'_{ij} - \lambda^\circ_{ij} \right) \left[s_1 \sum_{l=1}^{M_2} \left\{ \left(\beta'_{il} - \beta^\circ_{il} \right) + \left(\beta'_{jl} - \beta^\circ_{jl} \right) \right\} + t_1 \sum_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{M_1} \sum_{l=1}^{M_2} \left(\beta'_{kl} - \beta^\circ_{kl} \right) \right] \tag{6.67}$$

which when applying (6.48) is seen to vanish. Similarly S_6 is shown to vanish.

By applying the results of the previous chapter and using the hypotheses of this theorem we have that

$$S_1 = \left[\sum_{R_1} \left(\lambda_{ij} - \lambda_{ij} \right)^2 \right] (p_1 - 2q_1 + r_1) \tag{6.68}$$

From (5.106)

$$\int_0^\infty e^{\lambda x} S_1 \, dx = \left[\sum_{R_1} \left(\lambda'_{ij} - \lambda^\circ_{ij} \right)^2 \right] \cdot \int_0^\infty e^{\lambda x} \Phi\left(x; \alpha_{R_{M_2}} \right) \left(\left\{ \left[\left(\frac{M_1 - 1}{M_1 - 1} \right)^2 x^2 - \frac{2(M_1-1)}{M_1(M_1-2)} \right] F_2(x) + \frac{(M_1-1)}{M_1(M_1-2)} C(x) \right\} G_2(x) + \frac{2(M_1-1)}{(M_1-2)(M_1-3)} x \, G_3(x) \, F_3(x) \right) dx \tag{6.69}$$

which we must show is positive. The integration over the last two terms is clearly positive, so it is sufficient to verify that

$$\int_o^\infty e^{\lambda x}\,\Phi\left(x;\alpha_{R_{M_2}}\right)\left[\left(\frac{M_1-1}{M_1-2}\right)x^2-\frac{2}{M_1}\right]\;G_2(x)\;F_2(x)\;dx>0 \tag{6.70}$$

As in Theorem 5.5, integrate

$$\int_o^\infty e^{\lambda x}\;\Phi\left(x;\alpha_{R_{M_2}}\right)\;G_2(x)\;F_2(x)\;dx$$

by parts, which when substituted into (6.70) results in

$$\int_o^\infty e^{\lambda x}\;\Phi\left(x;\alpha_{R_{M_2}}\right)\left[\left(\frac{M_1-1}{M_1-2}\right)x^2-\frac{2}{M_1}\right]G_2(x)\;F_2(x)\;dx$$

$$=\left(\frac{1}{2}-\frac{2}{M_1}\right)\int_o^\infty e^{\lambda x}\;\Phi\left(x;\alpha_{R_{M_2}}\right)\;G_2(x)\;F_2(x)\;dx$$

$$+\frac{1}{2}\int_o^\infty e^{\lambda x}\;G_2(x)\left[\lambda x\,\Phi\left(x;\alpha_{R_{M_2}}\right)F_2(x)+\;p_{M_2}\left(x;\alpha_{R_{M_2}}\right)F_2(x)\right.$$

$$\left.+\,\Phi\left(x;\alpha_{R_{M_2}}\right)\;\frac{d\,F_2(x)}{dx}\right]\;dx \tag{6.71}$$

which is seen to be positive for $M_1>3$.

Similarly, the integral over S_3 is shown to be positive.

Finally, by again making use of (6.48), we have that

$$S_2=\sum_{R_2}\sum\left(\beta'_{ij}-\beta^{\circ}_{ij}\right)^2\;(v-w_1-w_2+u) \tag{6.72}$$

Now

$$u = \underbrace{\int_{-\infty}^{x}\cdots\int}_{M_1-2\text{ fold}} G\left(x, x, \xi_3, \ldots, \xi_{M_1}; 0; \alpha_{R_{M_1}}\right) d\xi_3 \ldots d\xi_{M_1}$$

$$\cdot \underbrace{\int_{-\infty}^{x}\cdots\int}_{M_2-2\text{ fold}} G\left(x, x, \eta_3, \ldots, \eta_{M_2}; 0; \alpha_{R_{M_2}}\right) d\eta_3 \ldots d\eta_{M_2}$$

$$= G_2\left(x;\alpha_{R_{M_1}}\right) F_2\left(x;\alpha_{R_{M_1}}\right) G_2\left(x;\alpha_{R_{M_1}}\right) F_2\left(x;\alpha_{R_{M_1}}\right) \tag{6.73}$$

where the notation that is introduced here is apparent, and

$$v = \underbrace{\int_{-\infty}^{x}\cdots\int}_{M_1-\text{fold}} \frac{\partial^2}{\partial \xi_1^2} G\left(\xi; 0; \alpha_{R_{M_1}}\right) d|\xi| \underbrace{\int_{-\infty}^{x}\cdots\int}_{M_2-\text{fold}} \frac{\partial^2}{\partial \eta_1^2} G\left(\eta; 0; \alpha_{R_{M_2}}\right) d|\eta| \tag{6.74}$$

Following the same technique used in partially evaluating q in Lemma 5. 6, introduce

$$\alpha_{\rho_1} = \begin{pmatrix} 1 & & \rho_1 \\ & \ddots & \\ \rho_1 & & 1 \end{pmatrix} \quad \text{and} \quad \alpha_{\rho_2} = \begin{pmatrix} 1 & & \rho_2 \\ & \ddots & \\ \rho_2 & & 1 \end{pmatrix}$$

as M_1-by-M_1 and M_2-by-M_2 matricies respectively. Then perform the necessary algebraic manipulations, and finally take limits as $\rho_1 \to \frac{-1}{M_1-1}$ and $\rho_2 \to \frac{-1}{M_2-1}$. In this way, v can be shown to be (after taking limits)

$$v = x^2 G_1\left(x;\alpha_{R_{M_1}}\right) F_1\left(x;\alpha_{R_{M_1}}\right) G_1\left(x;\alpha_{R_{M_2}}\right) F_1\left(x;\alpha_{R_{M_2}}\right)$$

$$- x\left[G_2\left(x;\alpha_{R_{M_1}}\right) F_2\left(x;\alpha_{R_{M_1}}\right) G_1\left(x;\alpha_{R_{M_2}}\right) F_1\left(x;\alpha_{R_{M_2}}\right)\right.$$

$$\left.+ G_2\left(x;\alpha_{R_{M_2}}\right) F_2\left(x;\alpha_{R_{M_2}}\right) G_1\left(x;\alpha_{R_{M_1}}\right) F_1\left(x;\alpha_{R_{M_1}}\right)\right]$$

$$+ G_2\left(x;\alpha_{R_{M_1}}\right) F_2\left(x;\alpha_{R_{M_1}}\right) G_2\left(x;\alpha_{R_{M_2}}\right) F_2\left(x;\alpha_{R_{M_2}}\right) \tag{6.75}$$

w_1 and w_2 can be similarly evaluated and shown to be (again after taking limits)

$$w_1 = -x\, G_2\left(x;\alpha_{R_{M_1}}\right) F_2\left(x;\alpha_{R_{M_1}}\right) G_1\left(x;\alpha_{R_{M_2}}\right) F_1\left(x;\alpha_{R_{M_2}}\right)$$

$$+ G_2\left(x;\alpha_{R_{M_1}}\right) F_2\left(x;\alpha_{R_{M_1}}\right) G_2\left(x;\alpha_{R_{M_2}}\right) F_2\left(x;\alpha_{R_{M_2}}\right)$$

and

$$w_2 = -x\, G_2\left(x;\alpha_{R_{M_2}}\right) F_2\left(x;\alpha_{R_{M_2}}\right) G_1\left(x;\alpha_{R_{M_1}}\right) F_1\left(x;\alpha_{R_{M_1}}\right)$$

$$+ G_2\left(x;\alpha_{R_{M_1}}\right) F_2\left(x;\alpha_{R_{M_1}}\right) G_2\left(x;\alpha_{R_{M_2}}\right) F_2\left(x;\alpha_{R_{M_2}}\right)$$

After substitution, it is immediate that

$$v - w_1 - w_2 + u = x^2 G_1\left(x;\alpha_{R_{M_1}}\right) F_1\left(x;\alpha_{R_{M_1}}\right) G_1\left(x;\alpha_{R_{M_2}}\right) F_1\left(x;\alpha_{R_{M_2}}\right)$$

and finally that

$$\int_0^\infty e^{\lambda x}\left[v - w_1 - w_2 + u\right] dx > 0 \tag{6.76}$$

Hence, $J(\lambda;\alpha_o)$ is a local minimum in Γ_2 or equivalently, $P_D(\lambda;\alpha_o)$ is a local maximum in Γ_2. QED

6.3 Choosing the Largest of Several Local Maxima

Several signal sets have been shown to be local maxima in the class of admissible α for which $D \le M-2$, i.e., Γ_2. For large signal-to-noise ratio at any rate the largest of these can be found.

Theorem 6.3: Of all the local maxima (i.e., all the α_o in the previous sections), at large signal-to-noise ratio the one which has the largest probability of detection is the one which satisfies the condition

$$|M_1 - M_2| \le 1 \tag{6.77}$$

Remark: Thus, for M even, the largest local maximum consists of two regular simplicies placed orthogonal to each other, each formed from M/2 signal vectors; and for M odd, one is formed from $\frac{M-1}{2}$ signals and one from $\frac{M+1}{2}$. For each M there is only one of the local maxima which satisfies (6.77). Hence, the solution is unique.

It should also be noted that in all of the optimization performed to date, the assumption of an equi-likely a priori distribution has been made. In the case of the regular simplex, however, the resultant probability of detection turned out to be independent of the a priori distribution. That is, even though the optimization was carried out for an equi-likely a priori distribution, the probability of detection for the regular simplex does not depend on the a priori distribution. This is also the case for the optimal signal sets obtained in this chapter if M is even. When M is odd, however, there is a dependence of the probability of detection on the a priori distribution, a dependence which decreases as M increases.

The independence of a priori distribution property occurs when the signal structure is uniform, that is, the array of signals when looking from a given signal vector is identical for all signals. This is clearly the case for the regular simplex. For two regular simplicies placed orthogonal to one another, a signal chosen from one of them is automatically uniform with respect to the other signal vectors in that simplex. However, it will not be uniform with respect to the signal vectors of the other regular simplex unless both simplicies have the same number of signals, which occurs, of course, when M is even. Equivalently stated, the region in which a signal is decided upon when the received vector falls into it has the same shape for all signals.

When the uniformity property is present, then

$$P_D = \sum_{i=1}^{M} P(S_i \text{ was transmitted})$$

$$\cdot\ P(S_i \text{ was decided upon} / S_i \text{ was transmitted})$$

$$= P(S_i \text{ was decided upon} / S_i \text{ was transmitted}) \cdot$$

$$\cdot \sum_{i=1}^{M} P(S_i \text{ was transmitted})$$

$$= P(S_i \text{ was decided upon} / S_i \text{ was transmitted}),$$

and is independent of i.

<u>Proof:</u> From Theorem 4.6, for any α, and for large λ

$$P_D(\lambda;\alpha) \approx 1 - \frac{1}{2M} \sum_{i>j}\sum \frac{1}{\sqrt{2\pi}\,\gamma_{ij}\,\lambda}\, e^{-\frac{1}{2}\gamma_{ij}^2 \lambda^2} \tag{4.79}$$

where

$$\gamma_{ij} = \sqrt{\frac{1-\lambda_{ij}}{1+\lambda_{ij}}} \tag{4.80}$$

Applying this result to the α_o in this chapter, we obtain

$$P_D(\lambda;\alpha_o) \quad 1 - \frac{1}{\sqrt{2\pi}\, 2M\lambda}\left[M_1(M - M_1)\, e^{-\frac{1}{2}\lambda^2} \right.$$

$$\left. + \frac{(M_1 - 1)\sqrt{M_1(M_1-2)}}{2} \exp\left\{-\tfrac{1}{2}\lambda^2\left(\frac{M_1}{M_1-2}\right)\right\} + \frac{(M-M_1-1)\sqrt{(M-M_1)(M-M_1-2)}}{2} \exp\left\{-\tfrac{1}{2}\lambda^2\left(\frac{M-M_1}{M-M_1-2}\right)\right\}\right] \tag{6.78}$$

where M_2 has been replaced by $M - M_1$. Assuming for the present that M_1 can vary continuously, it can be shown that when $M_1 = M/2$

$$\frac{\partial P_D(\lambda;\alpha_o)}{\partial M_1} = 0 \tag{6.79}$$

and

$$\left.\frac{\partial^2 P_D(\lambda;\alpha_o)}{\partial M_1^2}\right|_{M_1 = \frac{M}{2}} < 0 \tag{6.80}$$

Also, since $P_D(\lambda;\alpha)$, as approximated in (4.79), is strictly convex down as a function of the vector

$$\gamma = \begin{pmatrix} \gamma_{12} \\ \cdot \\ \cdot \\ \cdot \\ \gamma_{M-1,\,M} \end{pmatrix} \tag{6.81}$$

we have that the solution is unique and hence that M_1 should be chosen as that integer which is closest to $M/2$. This is equivalent to saying that M_1 and M_2 should satisfy

$$|M_1 - M_2| \leq 1$$

<u>QED</u>

6.4 The Example of 5 Signal Vectors in 3 Dimensions

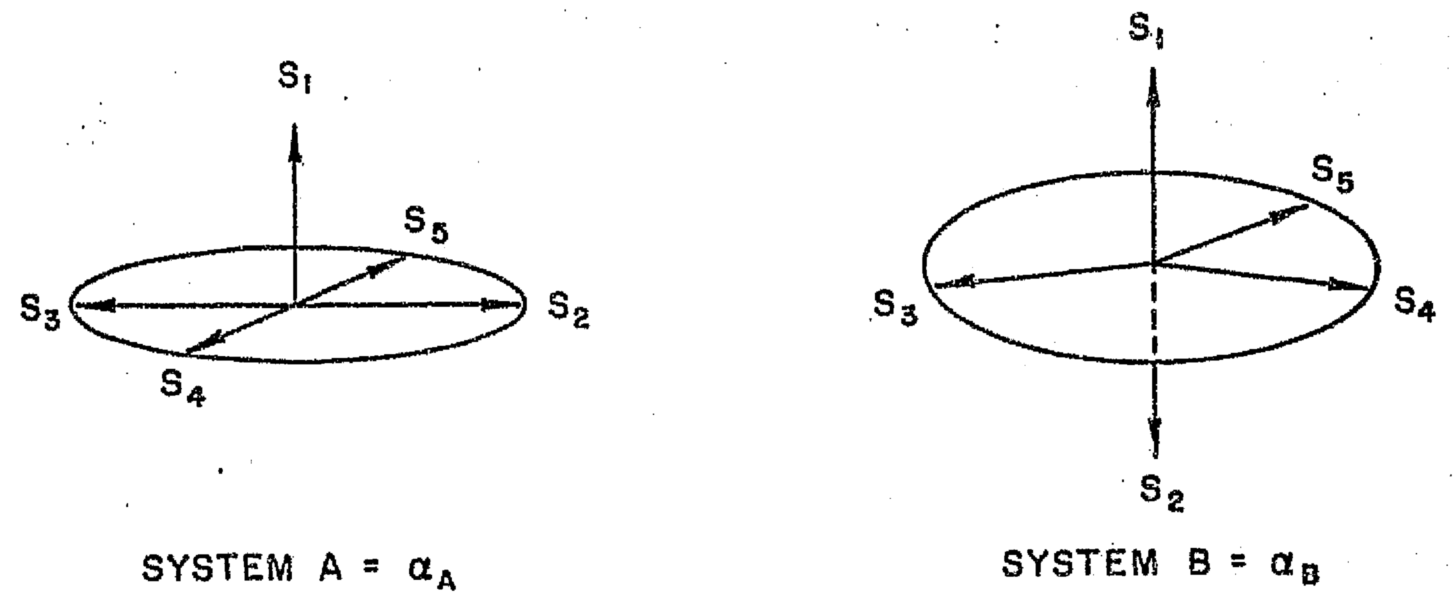
As an application of the results of this chapter, we shall consider the case of 5 points in 3 dimensions. Figure 6.3 is a pictorial diagram of the two systems to be compared; System B with 3-points on the equator and one on each pole is a special case of α_o, when M=5, M_1=3, and M_2=2, and is therefore a local maximum of $P_D(\lambda;\alpha)$.

Consider a small arbitrary perturbation of System A, and approximate it by

$$J\left|\lambda;\alpha(\theta)\right| \approx J(\lambda;\alpha_A) + \left[\sum_{i>j}\sum \frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{ij}} \left.\frac{\partial\tilde{\lambda}_{ij}(\theta)}{\partial\theta}\right|_{\theta=0}\right]\theta \tag{6.82}$$

for small θ, where

$$\tilde{\lambda}_{ij}(\theta) = S_i'(\theta)^* S_j'(\theta) \tag{6.83}$$



2 SYSTEMS OF 5 SIGNALS IN 3-DIMENSIONS

FIGURE 6.3

The coordinates of System A are, from Figure 6.4:

$$
\begin{aligned}
S_1 &= (0,\ 0,\ 1) \\
S_2 &= (0,\ 1,\ 0) \\
S_3 &= (0,\ -1,\ 0) \\
S_4 &= (1,\ 0,\ 0) \\
S_5 &= (-1,\ 0,\ 0). \qquad (6.84)
\end{aligned}
$$

The coordinates of $S_i'(\theta)$ can be expressed in terms of trigonometric functions from Figure 6.4 at $\theta = 1$, and are as follows:

$$
\begin{aligned}
S_1'(\theta) &= (\sin\theta\,\epsilon_1 \sin\theta\,\psi_1,\ \sin\theta\,\epsilon_1,\ \cos\theta\,\psi_1,\ \cos\theta\,\epsilon_1) \\
S_2'(\theta) &= (\cos\theta\,\epsilon_2 \sin\theta\,\psi_2,\ \cos\theta\,\epsilon_2,\ \cos\theta\,\psi_2,\ \sin\theta\,\epsilon_2) \\
S_3'(\theta) &= (-\cos\theta\,\epsilon_3 \sin\theta\,\psi_3,\ -\cos\theta\,\epsilon_3,\ \cos\theta\,\psi_3,\ \sin\theta\,\epsilon_3) \\
S_4'(\theta) &= (\cos\theta\,\epsilon_4 \cos\theta\,\psi_4,\ -\cos\theta\,\epsilon_4,\ \sin\theta\,\psi_4,\ \sin\theta\,\epsilon_4) \\
S_5'(\theta) &= (-\cos\theta\,\epsilon_5 \cos\theta\,\psi_5,\ \cos\theta\,\epsilon_5,\ \sin\theta\,\psi_5,\ \sin\theta\,\epsilon_5) \qquad (6.85)
\end{aligned}
$$

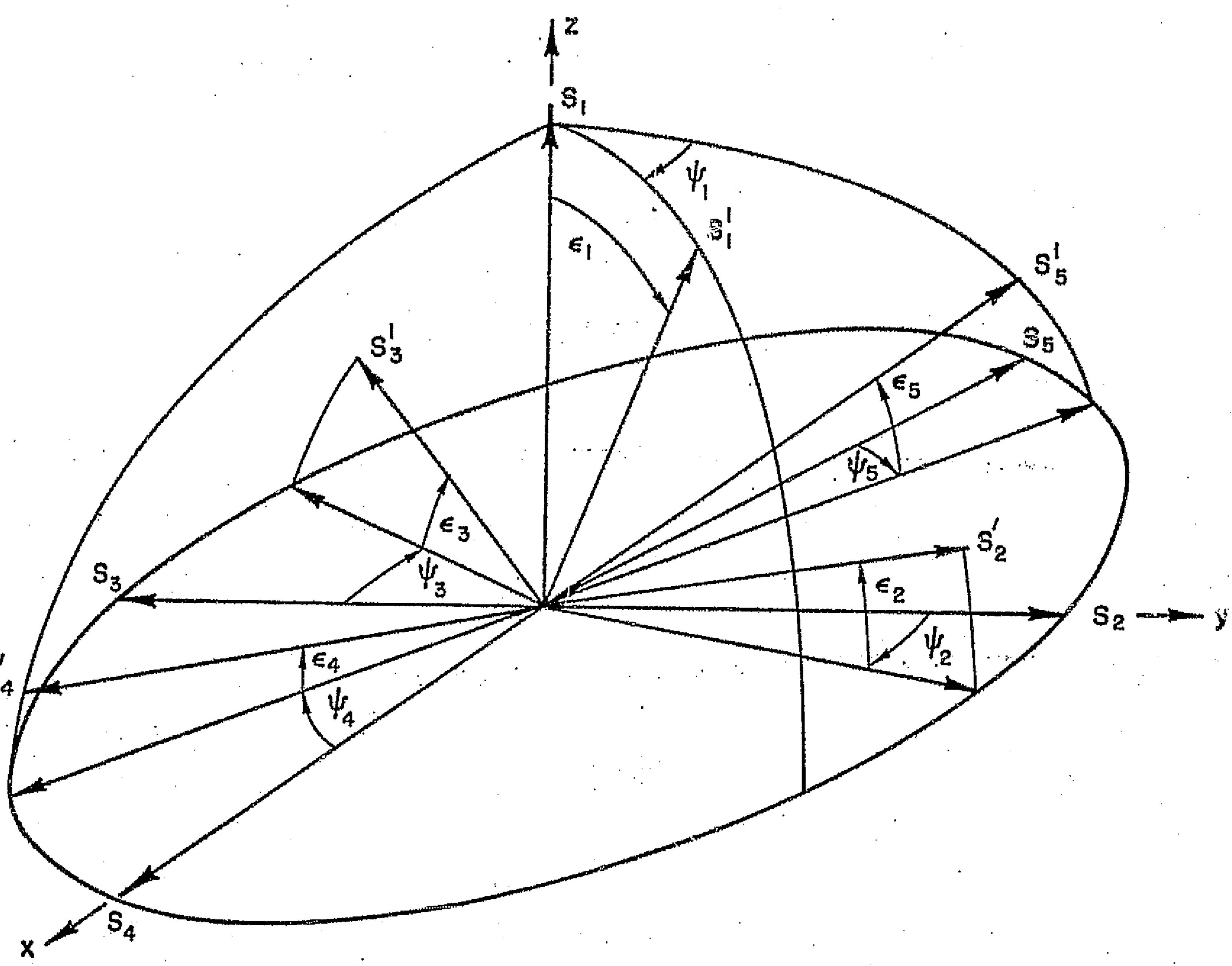
where the ϵ_i and the ψ_i are arbitrary perturbation angles as indicated in Figure 6.4.

By constructing the perturbation in this manner, the restriction of $D \leq 3$ is automatically satisfied.

Now,

$$
J(\lambda; \alpha_A) = \int_o^\infty dx\, e^{\lambda x} \left[\left\{ \int_{-\infty}^{x} G(\xi_1)\, d\xi_1 \right\} \cdot \right.
$$

$$
\left. \cdot \left\{ \int_{-\infty}^{x}\!\!\int G\left(\xi_2, \xi_3; 0; \alpha_{R_2}\right) d\xi_2 d\xi_3 \right\} \left\{ \int_{-\infty}^{x}\!\!\int G\left(\xi_4, \xi_5; 0; \alpha_{R_2}\right) d\xi_4\, d\xi_5 \right\} - \left\{ \phi(x) \right\}^5 \right]
$$



SYSTEM A - PERTURBED

FIGURE 6. 4

$$\frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{12}} = \frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{13}} = \frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{14}} = \frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{15}} = a$$

$$\frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{24}} = \frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{25}} = \frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{34}} = \frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{35}} = b$$

and

$$\frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{23}} = \frac{\partial J(\lambda;\alpha_A)}{\partial\lambda_{45}} = 0$$

Substituting

$$J\left(\lambda;\alpha(\theta)\right) \approx J(\lambda;\alpha_A)$$

$$+ \theta \; a \left(\sum_{j=2}^{5} \left. \frac{\partial\tilde{\lambda}_{1j}(\theta)}{\partial\theta} \right|_{\theta=0} \right)$$

$$+ b\,\theta \left\{ \left. \frac{\partial\tilde{\lambda}_{24}(\theta)}{\theta} \right|_{\theta=0} + \left. \frac{\partial\tilde{\lambda}_{25}(\theta)}{\partial\theta} \right|_{\theta=0} + \left. \frac{\partial\tilde{\lambda}_{34}(\theta)}{\partial\theta} \right|_{\theta=0} \left. \frac{\partial\tilde{\lambda}_{25}(\theta)}{\partial\theta} \right|_{\theta=0} \right\}$$

which can be shown to be

$$J\left(\lambda;\alpha(\theta)\right) \approx J(\lambda;\alpha_A) + a(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5)\,\theta\,.$$

Since the ϵ_i are arbitrary, they can be chosen to either increase or decrease $J\left(\lambda;\alpha(\theta)\right)$ in the neighborhood of $\theta = 0$. Therefore, not only is System A (consisting of 4 points on the equator and one on a pole) not a local maximum, but also not even an extremum. Figure 6.5 is a plot of the probability of error for these two systems, indicating the preference of System B to System A at all signal-to-noise ratio.

Further, it should be pointed out that the minimum distance between the different vectors is the same for both of these systems, thus proving that the relationship between maximizing the probability of detection and maximizing the minimum distance which exists when there is no dimensionality restriction does not exist when there are restrictions on the dimensionality more severe than $D \leq M-1$.